Towards a 2.5D Geometric Model in Mould Filling Simulation

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Abstract

Resin Infusion (RI) process is frequently used for large composite parts production. This Liquid Composite Molding method uses vacuum pressure to shape a plastic bag as a counter mold. Once a complete vacuum is achieved, the resin is sucked into a dry preform textile laminate via placed tubing. In this note we introduce a 2.5D model for a Liquid Composite Molding LCM process starting from a recent one introduced by Besson and Poussin in [5] for the Resin Transfer Molding RTM process. Moreover for 2.5D models defined over quadrilateral reference domains, we show the effectiveness of the use of the Proper Generalized Decomposition. Finally, we propose a procedure, for a particular class of 2.5D model, in order to perform a numerical simulation of a mould filling process.

Keywords: Liquid Composite Molding, Resin Transfer Molding, Proper Generalized Decomposition, Stokes equation. Darcy’s equation.

1 Introduction

Advanced fiber-reinforced composite materials have found important applications in the automotive and aerospace industry. For instance, Ford Motor Company makes automobile hoods from such composites, and aerospace companies like Airbus and Boeing use them to produce nose cones, wings, and tail pieces. Such composites are desirable because they are lightweight, high strength, offer design flexibility, and can be mass produced. For manufacturing of composite parts with a high content of oriented reinforcement fibers a lot of processes can be used. Liquid composite molding (LCM) is one of the methods for the manufacture of such composites and combines cost savings (over, for instance, hand-lay-up methods) with substantial performance improvements. But the impregnation of a dry preform with a liquid matrix by liquid composite moulding processes offers a very high potential for economical manufacturing of high performance composite components.

All liquid composite molding processes require that the resin injected into the mold is a reactive liquid. Some resins such as epoxy and urethane are highly reactive and must be kept separate until just before they are injected into the mold. Other resins are activated by a catalyst in the holding tank. These multi part resin systems require complex mixing, metering, and use of injection equipment with accurate ratio control. The multi part resin systems also may require heating tanks, hoses, pipes, and pumps; motionless mixing; efficient circulation to help prevent cure or degradation of the resin in a holding tank; and easy and safe cleaning/purging.

Resin transfer molding (RTM) is a process for the manufacture of fiber-reinforced composites. In recent years, liquid composites molding processes (LCM) and, particularly, resin transfer molding (RTM), are considered more and more in industrial sectors like the aerospace and automotive industries. An interesting
feature of the RTM process is its potential to become a high-speed process for the manufacturing of components at a low cost, giving cost. In the RTM process dry fiber reinforcement is preshaped and oriented into a skeleton known as the preform, which is inserted into a mold cavity which has the shape of the desired part. The mold is then closed and a pressurized low viscosity reactive fluid is injected into the mold cavity. During this stage, known as the injection phase, the resin impregnates and wets out the fibers. The curing reaction begins when the prepolymer is mixed just before the injection into the mold. During the curing reaction the resin polymerizes to become rigid plastic. The filling process in RTM can be investigated by using numerical simulation and/or experimental analysis. Because the experimental analysis is often expensive and time-consuming, the numerical simulation seems to be an appealing and efficient tool to analyze the mold filling stage, and it allows to reduce significantly the number of experimental tests needed.

It is beyond the scope of this paper to give a detailed overview of the extensive engineering literature on injection moulding. Here, we cite only the monographs [16] and [2].

Although this manufacturing technique appear to be simple, there is a complex interaction between operating and geometrical conditions as well as material properties. The physical processes of this technique to be simulated are characterized by the flow of the plastic melt, the a priori unknown free boundary between melt and escaping air (flow front), the energy balance containing convection, diffusion and viscous heating and, finally, by the rheological behaviour of the melt. The coupled phenomena of fluid flow, heat transfer and rheology during the filling stage of the cavity determine to a large extent the final properties of the moulded plastic parts. Clearly, this complexity of interaction motivates the mathematical modeling.

Despite a continuing fall in the cost of computer power which brings into sight the feasibility of a full three-dimensional filling analysis, numerical simulation of the non-Newtonian Navier-Stokes equations, including the free boundary/surface in geometrically complex, spatially three-dimensional regions, represents a formidable problem. Due to the geometrical feature of many plastic parts that one characteristic dimension (the thickness) is much smaller than the others, application of the Hele-Shaw (lubrication) approximation can be used with an accuracy satisfactory for most parts. This reduces the complexity of the governing equations leading to a drastic reduction in the demand on computer power.

The aim of this note is two fold. First we propose a 2.5D LCM dimensional model, which provides a mathematical justification to some of our numerical simulations performed over this model in some our previous research [14], [12], [10], [6], [15]. The starting point is the use of a mathematical model for the RTM process previously introduced by Besson and Pousin [5]. In particular, Besson and Pousin [5] derive a set of Helle-Shaw like equations. These equations can be seen as a general 2.5D dimensional approach of the original model. Our contribution is to derive a Darcy like equation under the constraint of a two dimensional behavior of the viscosity term. This constraint is no so far to the standard assumptions used in order to simulate mould filling processes in real life problems. The second goal of the present work is to justify the use in this framework of the Proper Generalized Decomposition (PGD) due to the use of this novel approach in the simulation of mould filling process [1]. To this end we will justify the use of this technique in the approximating 2.5D model. Finally, we give a model for the classical quadrilateral mould, in order to give a procedure to simulate the filling process.

The paper is organized as follows. In Section 2 we introduce the mathematical model for LCM proposed by Besson and Pousin [5]. Next, in Section 3 we give the 2.5D model and derive a Darcy like equation related with this model. Next, under the hypothesis of a quadrilateral two dimensional reference domain we justify the use of the Proper Generalized Decomposition PGD. Finally, in Section 4 we give a procedure to simulate the filling process for a particular class of moulds.

2 A mathematical model for LCM

The underlying mathematical equations describing the physical phenomena in many engineering applications (e.g., injection moulding) are the result of applying continuum mechanics techniques. These consist of a combination of general conservation laws and equations of state characteristic for the particular medium. The basic principles governing the motion of a viscous, incompressible fluid are conservation of mass and momentum (Navier-Stokes equations) together with a constitutive equation. We restrict ourselves to considering the filling phase in injection moulding. Thus we assume that the fluid is incompressible.

In order to model resin flow on the next length scale, the flow characteristics of the air phase need to be taken into account. Two- phase flow models, which take into account the interaction between resin, air, and
fiber, are a natural vehicle. Such models describe the displacement, within the fiber preform, of air by the resin. We assume that air and resin do not mix, i.e., remain separate phases.

To describe this let us consider $W \subset \mathbb{R}^3$, the mould, be a bounded open set with sufficiently regular boundary that it can be represented as follows. Let $\Omega \subset \mathbb{R}^3$ be a two dimensional reference domain (i.e. an open and connected set) such that, without loss of generality, we may assume that 

$$\Omega \subset \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \xi_3 = 0\} \cong \mathbb{R}^2.$$

From now, we will identify $\Omega$ with an open connected set in $\mathbb{R}^2$. Assume the existence of two Lipschitz functions $h_\alpha : \overline{\Omega} \to \mathbb{R}$ for $\alpha = \{-, +\}$ such that

$$W = \{\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : (\xi_1, \xi_2) \in \Omega \text{ and } h_-(\xi_1, \xi_2) < \xi_3 < h_+(\xi_1, \xi_2)\}.$$

The mould represented by the bounded domain $W$ in $\mathbb{R}^3$ is decomposed into two subdomains $W_i(t)$ for $i = 1, 2$. These domains are separated by an interface, denoted by $S(t)$. The domain $W_1(t)$ represents the resin part and $W_2(t)$ the air. Let $T$ be the filling time of the domain and assume that the interface $S(t)$ remains a surface sufficiently regular and connected. Moreover, the domain $W$ has a piecewise $C^1$ boundary.

The fluid flow is driven by the Stokes equations. In order to represent the time evolution of the domains $W_1(t)$ and $W_2(t)$, a pseudo-concentration function $s(t, \xi)$ will be used. This function is given by $s(t, \xi) = \chi_{W_3(t)}(\xi)$ where

$$\chi_A(\xi) := \begin{cases} 1 & \text{if } \xi \in A, \\ 0 & \text{otherwise}, \end{cases}$$

is the indicator function of the set $A$. Then the free boundary $S(t)$ is fully described by the function $s$.

For $T > 0$ and $i = 1, 2, 3$ set the strictly positive functions

$$\mu_i : [-a, a] \times (0, T) \times W \to [m_{\mu}, M_{\mu}], \quad (\alpha, t, \xi) \mapsto \mu_i(z, t, \xi)$$

with $\mu_i(s(t, \xi), t, \xi) = \mu_\alpha^i(t, \xi)$ for $s(t, \xi) = 0$ and $\mu_i(s(t, \xi), t, \xi) = \mu_\alpha^i(t, \xi)$ for $s(t, \xi) = 1$, where $\mu^a$ denotes the air viscosity and $\mu^p$ the polymer viscosity, and

$$\mu(\alpha, t, \xi) = \begin{bmatrix} \mu_1(\alpha, t, \xi) & 0 & 0 \\ 0 & \mu_2(\alpha, t, \xi) & 0 \\ 0 & 0 & \mu_3(\alpha, t, \xi) \end{bmatrix}$$

is a diagonal anisotropic viscosity tensor. Finally, $G_e \subset \partial W$ denotes the inlet set and $G_s \subset \partial W$ the outlet set. In order to simplify we assume that both sets are not vertical, that is, there exists two closed sets $\Omega^e, \Omega^s \subset \Omega$ such that $\Omega^e \cap \Omega^s = \emptyset$,

$$G_e = \{(\xi_1, \xi_2, h_+(\xi_1, \xi_2)) : (\xi_1, \xi_2) \in \Omega^e\},$$

where

$$\partial_{\xi_1} h_+ = \partial_{\xi_2} h_+ = 0 \text{ for } (\xi_1, \xi_2) \in \Omega^e \quad (2.1)$$

and

$$G_s = \{(\xi_1, \xi_2, h_-(\xi_1, \xi_2)) : (\xi_1, \xi_2) \in \Omega^s\},$$

where

$$\partial_{\xi_1} h_+ = \partial_{\xi_2} h_- = 0 \text{ for } (\xi_1, \xi_2) \in \Omega^s. \quad (2.2)$$

We illustrate the construction of a mould with the following useful example.
Example 2.1 Let us consider $\Omega_\delta = [-\delta, 1+\delta][0,1] \subset \mathbb{R}^3$; for some fixed $\delta > 0$. In consequence, we can identify $\Omega_\delta$ with the open set $[-\delta, 1+\delta][0,1+\delta] \subset \mathbb{R}^2$. We remark that $\Omega_\delta = (0,1] \cup \bigcup_{i=1}^{4} \Delta_i$, where

$$
\Delta_1 = \{(\xi_1, \xi_2) : -\delta < \xi_2 \leq 0 \text{ and } \xi_2 \leq \xi_1 \leq 1 - \xi_2\}, \\
\Delta_2 = \{(\xi_1, \xi_2) : 1 \leq \xi_1 < 1 + \delta \text{ and } 1 - \xi_1 \leq \xi_2 \leq \xi_1\}, \\
\Delta_3 = \{(\xi_1, \xi_2) : 1 \leq \xi_2 < 1 + \delta \text{ and } 1 - \xi_2 \leq \xi_2 \leq \xi_1\}, \\
\Delta_4 = \{(\xi_1, \xi_2) : -\delta < \xi_1 \leq 0 \text{ and } \xi_1 \leq \xi_2 \leq 1 - \xi_1\},
$$

(see Figure 2). Then we define $h_+, h_- : \Omega_\delta \rightarrow \mathbb{R}$ as follows

$$
h_+(\xi_1, \xi_2) = \begin{cases}
\delta & \text{if } (\xi_1, \xi_2) \in [0,1] \times [0,1] \\
\xi_2 + \delta & \text{if } (\xi_1, \xi_2) \in \Delta_1 \\
(1 + \delta) - \xi_1 & \text{if } (\xi_1, \xi_2) \in \Delta_2 \\
(1 + \delta) - \xi_2 & \text{if } (\xi_1, \xi_2) \in \Delta_3 \\
\xi_1 + \delta & \text{if } (\xi_1, \xi_2) \in \Delta_4
\end{cases}
$$

$$
h_-(\xi_1, \xi_2) = -h_+(\xi_1, \xi_2).
$$

We remark that the maps $h_+$ and $h_-$ can be easily extended to $\overline{\Omega_\delta}$ taking $h_+|_{\partial \Omega_\delta} = -h_-|_{\partial \Omega_\delta} = 0$. Finally, we consider the mould

$$W_\delta = \{(\xi) = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : (\xi_1, \xi_2) \in \Omega_\delta \text{ and } h_-(\xi_1, \xi_2) < \xi_3 < h_+(\xi_1, \xi_2)\},$$

where the inlet and outlet sets are defined by using

$$\Omega_\delta^e = \{(\xi_1, \xi_2) \in \Omega_\delta : 0 \leq \xi_1 \leq \delta\} \text{ and } \Omega_\delta^w = \{(\xi_1, \xi_2) \in \Omega_\delta : 1 - \delta \leq \xi_1 \leq 1\},$$

respectively, that is,

$$G_e = \{(\xi_1, \xi_2, \delta) : (\xi_1, \xi_2) \in \Omega_\delta^e\} \text{ and } G_w = \{(\xi_1, \xi_2, -\delta) : (\xi_1, \xi_2) \in \Omega_\delta^w\}.$$ 

Since $\Omega_\delta^e, \Omega_\delta^w \subset [0,1] \times [0,1]$ and $h_+|_{\Omega_\delta^w} = \delta$ (respectively, $h_-|_{\Omega_\delta^e} = -\delta$) on $[0,1] \times [0,1]$ conditions (2.1) and (2.2) hold.

2.1 The equations of the LCM model

With the previous notations, the air and polymer flows are driven by the following anisotropic equations

$$\text{div}(\mu \nabla v) + \nabla p = f \text{ in } W$$

$$\text{div} v = 0 \text{ in } W$$

$$\partial_t s + v \cdot \nabla s = 0 \text{ in } (0, T) \times W,$$

here the inertial term is neglected compared with viscous terms, $v = v(\xi) = (v_1(\xi), v_2(\xi), v_3(\xi))$ represents the fluid velocity, $p = p(\xi)$ is the pressure, $f = (0, 0, -pg)$ is the gravity force, $s = s(t, \xi)$ is the pseudo-concentration function such that $\mu$ is the diagonal turbulent viscosity tensor with $\mu_i = \mu_i(s(t, \xi), t, \xi)$ for $i = 1, 2, 3$. In particular, (2.3)-(2.5) can be written

$$-\sum_{i=1}^{3} \partial_{\xi_i} (\mu_i \partial_{\xi_i} v_j) + \partial_{\xi_j} p = f_j \quad \text{for } j = 1, 2, 3, \text{ in } W$$

$$\sum_{i=1}^{3} \partial_{\xi_i} v_i = 0 \text{ in } W;$$

$$\partial_t s + \sum_{i=1}^{3} v_i \partial_{\xi_i} s = 0 \text{ in } (0, T) \times W,$$
respectively. Moreover, the boundary conditions for the velocity \( v \) and the pseudo-concentration function \( s \) are

\[
\begin{align*}
v &= 0 \text{ in } \partial W \setminus (G_e \cup G_s) \quad (2.6) \\
v &= v_e \text{ on } G_e \quad (2.7) \\
v &= v_s \text{ on } G_s \quad (2.8) \\
s(t, \xi) &= 1 \text{ on } (0, T) \times G_e \quad (2.9) \\
s(0, \xi) &= 0 \text{ for all } \xi \in W. \quad (2.10)
\end{align*}
\]

Recall that the pseudoconcentration function is used to represent the time evolution of the resin part and its complementary the air part. Then (2.9) says us that for all time \( t \) the inlet set is full of resin and (2.10) implies that at time \( t = 0 \) the mould is full of air. We assume that the velocity \( v \) satisfies \( v_e \cdot n_+ > 0 \) on \( G_e \) and \( v_s \cdot n_- < 0 \) on \( G_s \), here

\[
n_+ = \frac{(\partial x_1 h_+, \partial x_2 h_+, -1)}{\sqrt{1 + (\partial x_1 h_+)^2 + (\partial x_2 h_+)^2}} \quad \text{for } (x_1, x_2) \in \Omega^e,
\]

and

\[
n_- = \frac{(\partial x_1 h_-, \partial x_2 h_-, -1)}{\sqrt{1 + (\partial x_1 h_-)^2 + (\partial x_2 h_-)^2}} \quad \text{for } (x_1, x_2) \in \Omega^s.
\]

Next we introduce the following two measures, let be

\[
d := \sup \{ ||\xi - \eta|| : \xi, \eta \in \partial W \setminus (G_e \cup G_s) \},
\]

the diameter of \( \partial W \setminus (G_e \cup G_s) \) and

\[
h := \sup \{ |h_+(\xi_1, \xi_2) - h_-(\xi_1, \xi_2)| : (\xi_1, \xi_2) \in \Omega \},
\]

5
We assume that $d \gg h$ and hence we introduce the ratio
\[ \epsilon := \frac{h}{d}, \]
clearly, $\epsilon \to 0$ if and only if $h \to 0$. Without loss of generality we may assume that $d = 1$. Then in order to make the mould $W$ independent of $\epsilon$ we take the following change of variables of $\xi$ and $u$
\[ x_1 = \xi_1, \quad x_2 = \xi_2, \quad x_3 = \xi_3/\epsilon = \xi_3/h, \]
\[ u_1 = v_1, \quad u_2 = v_2, \quad u_3 = v_3/\epsilon = v_3/h. \]
and since the pressure $p$ remains large we take
\[ p = \frac{q}{\epsilon^2} = \frac{q}{h^2}. \quad (2.11) \]
Then we introduce
\[ W' := \left\{ (x_1, x_2, x_3) : \frac{h_-(x_1, x_2)}{h} < x_3 < \frac{h_-(x_1, x_2)}{h} \text{ and } (x_1, x_2) \in \Omega \right\}, \]
\[ G'_t := \{ x = x(\xi) : \xi \in G_t \}, \]
\[ G'_s := \{ x = x(\xi) : \xi \in G_s \}. \]
where $x = (x_1, x_2, x_3)$. Introduce $z_+(x_1, x_2) = h_+(x_1, x_2)/h$ and $z_-(x_1, x_2) = h_-(x_1, x_2)/h$, then
\[ G'_t = \{(x_1, x_2, z_+(x_1, x_2)) : (x_1, x_2) \in \Omega' \} \]
and
\[ G'_s = \{(x_1, x_2, z_-(x_1, x_2)) : (x_1, x_2) \in \Omega' \} \]
With these changes we obtain the following problem
\[
- \left[ 2 \sum_{i=1}^{2} \partial_{x_i} (\mu_t \partial_{x_i} u_j) \right] - \frac{1}{\epsilon^2} \partial_{x_3} (\mu_3 \partial_{x_3} u_j) + \frac{1}{\epsilon^2} \partial_{x_j} q = 0 \quad \text{for } j = 1, 2, \text{ in } W' \\
- \left[ 2 \sum_{i=1}^{2} \partial_{x_i} (\epsilon \mu_t \partial_{x_i} u_3) \right] - \frac{1}{\epsilon} \partial_{x_3} (\mu_3 \partial_{x_3} u_3) + \frac{1}{\epsilon} \partial_{x_3} q = -\rho g \quad \text{in } W' \\
\sum_{i=1}^{3} \partial_{x_i} u_i = 0 \text{ in } W', \\
\partial_t s + \sum_{i=1}^{3} u_i \partial_{x_i} s = 0 \text{ in } (0, T) \times W',
\]
that is,
\[
- \left[ 2 \sum_{i=1}^{2} \partial_{x_i} (\mu_t \partial_{x_i} u_j) \right] - \partial_{x_3} (\mu_3 \partial_{x_3} u_j) + \partial_{x_j} q = 0 \quad \text{for } j = 1, 2, \text{ in } W' \quad (2.12) \\
- \epsilon^2 \sum_{i=1}^{3} \partial_{x_i} (\mu_t \partial_{x_i} u_3) + \partial_{x_3} q = -\epsilon^3 \rho g \quad \text{in } W' \quad (2.13) \\
\sum_{i=1}^{3} \partial_{x_i} u_i = 0 \text{ in } W', \quad (2.14) \\
\partial_t s + \sum_{i=1}^{3} u_i \partial_{x_i} s = 0 \text{ in } (0, T) \times W', \quad (2.15)
\]
with the boundary conditions

\[ u = 0 \text{ in } \partial W' \setminus (G_e' \cup G_s'), \quad (2.16) \]
\[ u = u_e \text{ in } G_e', \quad (2.17) \]
\[ u = u_s \text{ in } G_s', \quad (2.18) \]
\[ s(t, x) = 1 \text{ on } (0, T) \times G_e', \quad (2.19) \]
\[ s(0, x) = 0 \text{ for all } x \in W'. \quad (2.20) \]

Observe, that taking \( \epsilon = 0 \) we have the problem

\[ -\partial_{x_3} (\mu_3 \partial_{x_3} u_1) + \partial_{x_3} q = 0 \text{ in } W', \quad (2.21) \]
\[ -\partial_{x_3} (\mu_3 \partial_{x_3} u_2) + \partial_{x_3} q = 0 \text{ in } W', \quad (2.22) \]
\[ \partial_{x_3} q = 0 \text{ in } W', \quad (2.23) \]
\[ \sum_{i=1}^{3} \partial_{x_i} u_i = 0 \text{ in } W', \quad (2.24) \]
\[ \partial_t s + \sum_{i=1}^{3} u_i \partial_{x_i} s = 0 \text{ in } (0, T) \times W'. \quad (2.25) \]

with the boundary and initial conditions (2.16)-(2.20). These equations are called the Helle-Shaw equations. These equations can be modified in order to obtain a simplified 2.5D problem for the velocity \( u \). Moreover, in [5] the following useful result is proved,

**Theorem 2.2** There exists a Hilbert space \( \mathcal{H} \) such that:

(a) There exists on \( \mathcal{H} \) a solution \((u, q, s)\) of the variational problem associated with (2.21)-(2.25) with the boundary and initial conditions (2.16)-(2.20), and

(b) For any \( \epsilon > 0 \) there exists \((u_\epsilon, q_\epsilon, s_\epsilon)\) \(\in\mathcal{H}\) a solution of the variational problem associated with (2.12)-(2.15) with the boundary and initial conditions (2.16)-(2.20).

Moreover, \((u_\epsilon, q_\epsilon, s_\epsilon)\) weakly converges to \((u, q, s)\) as \(\epsilon \to 0\), in \(\mathcal{H}\).

### 3 Towards a 2.5D mould filling model and the Proper Generalized Decomposition

In this section we use the fact that the model (2.21)-(2.24) is the weak limit of the original LCM model given by (2.3)-(2.4) to propose a 2.5D approximating model of (2.3)-(2.4). To this end, our starting point are the equations (2.21)-(2.24). Now in order to obtain a more tractable problem, we assume that \(h_+\) and \(h_-\) satisfy

\[ h_+(x_1, x_2) = h_-(x_1, x_2) = 0 \text{ for all } (x_1, x_2) \in \partial \Omega, \]

(see Example 2.1). In particular, the velocity vectors \(u_e\) and \(u_s\) in (2.17) and (2.18) can be written as

\[ u_e = u(x_1, x_2, z_+(x_1, x_2)) = (u_1^e, u_2^e, u_3^e) \text{ for } (x_1, x_2) \in \Omega^e \]

and

\[ u_s = u(x_1, x_2, z_-(x_1, x_2)) = (u_1^s, u_2^s, u_3^s) \text{ for } (x_1, x_2) \in \Omega^s, \]

respectively, and where we assume that \(u_1^e, u_2^s : \Omega \to \mathbb{R}\) satisfy \(u_1^e|_{\partial \Omega} = u_2^s|_{\partial \Omega} = 0\) for \(1 \leq i \leq 3\). Since \(\Omega^e \cap \Omega^s = \emptyset\), it allows us to write the boundary conditions (2.16)-(2.18) as

\[ u(x_1, x_2, z_+(x_1, x_2)) = (u_1^e(x_1, x_2), u_2^e(x_1, x_2), u_3^e(x_1, x_2)) \neq 0 \]
if and only if \((x_1, x_2) \in \Omega^e\), and

\[
u(x_1, x_2, z_-(x_1, x_2)) = (u_1^e(x_1, x_2), u_2^e(x_1, x_2), u_3^e(x_1, x_2)) \neq 0
\]

if and only if \((x_1, x_2) \in \Omega^c\). Clearly \(u = 0\) if \((x_1, x_2) \in \Omega \setminus (\Omega^e \cup \Omega^c)\). On the other hand, (2.23) shows that \(q\) is independent of \(x_3\). It allows us to prove the following theorem. As usual \(H^s(W')\) (respectively, \(H^s(\Omega)\)) denotes the Sobolev space of order \(s\).

**Theorem 3.1** Assume that

\[\sigma = \sigma(s(t, x_1, x_2), x_1, x_2) = \frac{1}{\mu^3} > 0.\]

If \((\nu(x_1, x_2, x_3), q(x_1, x_2)) \in H^2(W')^3 \times H^1(\Omega)\) is a solution of (2.21)--(2.24) with the boundary and initial conditions (2.16)–(2.18), take

\[\pi_i = \pi_i(x_1, x_2) = -\frac{1}{2} \left( \frac{(z_+ - z_-)^2}{6} \cdot \sigma \partial_z q + u_i^e + u_i^s \right),\]

for \(i = 1, 2\). Then \((\pi_1(x_1, x_2), \pi_2(x_1, x_2))\) satisfy the equation

\[-[\partial_{z_1} ((z_+ - z_-)\pi_1) + \partial_{z_2} ((z_+ - z_-)\pi_2)] = u_3^e - u_3^s,\]

(3.1) for \((x_1, x_2) \in \Omega\) and \(\pi_1|_{\partial \Omega} = \pi_2|_{\partial \Omega} = 0\). Moreover,

\[u_i(x_1, x_2, x_3) = \frac{\sigma}{2} \cdot (x_3 - z_-)(x_3 - z_+)\partial_{z_3} q + u_i^e + u_i^s \frac{(x_3 - z_-)}{(z_+ - z_-)} - u_i^s \frac{(x_3 - z_+)}{(z_+ - z_-)},\]

(3.2) for \(i = 1, 2\) and

\[u_3(x_1, x_2, x_3) = - \left( \partial_{z_1} \left( \frac{\sigma}{2} \cdot \partial_{z_1} q + \partial_{z_2} \left( \frac{\sigma}{2} \cdot \partial_{z_2} q \right) \right) \left[ \frac{1}{2}(x_3 - z_-)^2(x_3 - z_-) - \frac{1}{6}(x_3 - z_+)^3 \right] + \frac{\sigma}{2} \cdot (\partial_{z_1} q \partial_{z_1} z_+ + \partial_{z_2} q \partial_{z_2} z_-) \left[ \frac{1}{2}(x_3 - z_+)^2 \right] + \frac{\sigma}{2} \cdot \partial_{z_1} q \partial_{z_2} z_+ + \partial_{z_2} q \partial_{z_2} z_+ \right) \left[ \frac{1}{2}(x_3 - z_-)^2 \right] - \left[ z_+^3 \partial_{z_1} \left( \frac{u_3^1 - u_1^1}{z_+ - z_-} \right) + z_3 \partial_{z_1} \left( \frac{u_3^1 z_+ - u_1^1 z_-}{z_+ - z_-} \right) \right] - \left[ z_+^3 \partial_{z_2} \left( \frac{u_3^2 - u_2^1}{z_+ - z_-} \right) + z_3 \partial_{z_2} \left( \frac{u_3^2 z_+ - u_2^1 z_-}{z_+ - z_-} \right) \right] - \kappa(x_1, x_2),\]

where \(\kappa\) is given by

\[
\kappa = \frac{\sigma}{2} \left( \partial_{z_1} q \partial_{z_2} z_+ + \partial_{z_2} q \partial_{z_2} z_+ \right) \left[ \frac{1}{2}(z_+ - z_-)^2 \right] - \frac{1}{2} \left[ \partial_{z_1} \left( \frac{u_3^1 - u_1^1}{z_+ - z_-} \right) + \partial_{z_2} \left( \frac{u_3^2 - u_2^1}{z_+ - z_-} \right) \right] z_+^2 - \left[ \partial_{z_1} \left( \frac{u_3^1 z_+ - u_1^1 z_-}{z_+ - z_-} \right) + \partial_{z_2} \left( \frac{u_3^2 z_+ - u_2^1 z_-}{z_+ - z_-} \right) \right] z_+ - \frac{1}{2} \left[ \partial_{z_1} \left( \frac{u_3^1 - u_1^1}{z_+ - z_-} \right) + \partial_{z_2} \left( \frac{u_3^2 - u_2^1}{z_+ - z_-} \right) \right] z_+^3 - \frac{1}{2} \left[ \partial_{z_1} \left( \frac{u_3^1 - u_1^1}{z_+ - z_-} \right) + \partial_{z_2} \left( \frac{u_3^2 - u_2^1}{z_+ - z_-} \right) \right] z_+^3 - \frac{1}{2} \left[ \partial_{z_1} \left( \frac{u_3^1 - u_1^1}{z_+ - z_-} \right) + \partial_{z_2} \left( \frac{u_3^2 - u_2^1}{z_+ - z_-} \right) \right] z_+^3 - \frac{1}{2} \left[ \partial_{z_1} \left( \frac{u_3^1 - u_1^1}{z_+ - z_-} \right) + \partial_{z_2} \left( \frac{u_3^2 - u_2^1}{z_+ - z_-} \right) \right] z_+^3 - \frac{1}{2} \left[ \partial_{z_1} \left( \frac{u_3^1 - u_1^1}{z_+ - z_-} \right) + \partial_{z_2} \left( \frac{u_3^2 - u_2^1}{z_+ - z_-} \right) \right] z_+^3 - \frac{1}{2} \left[ \partial_{z_1} \left( \frac{u_3^1 - u_1^1}{z_+ - z_-} \right) + \partial_{z_2} \left( \frac{u_3^2 - u_2^1}{z_+ - z_-} \right) \right] z_+^3.
\]

**Proof.** First at all (3.2) is obtained integrating (2.21)–(2.22). To obtain (3.2) we take into account that \(u_i\) must be a quadratic map on \(x_3\) and that \(u_i(x_1, x_2, z_-(x_1, x_2)) = u_i^e\) and \(u_i(x_1, x_2, z_-(x_1, x_2)) = u_i^s\) for \(i = 1, 2\). Observe that

\[
\int_{z_-}^{z_+} u_1(x) dx = (z_+ - z_-) \left( -\frac{(z_+ - z_-)^2}{6} \cdot \sigma \partial_z q + \frac{1}{2}(u_1^e + u_1^s) \right) = (z_+ - z_-) \pi_1(x_1, x_2),
\]

(3.4)

\[
\int_{z_-}^{z_+} u_2(x) dx = (z_+ - z_-) \left( -\frac{(z_+ - z_-)^2}{6} \cdot \sigma \partial_z q + \frac{1}{2}(u_2^e + u_2^s) \right) = (z_+ - z_-) \pi_2(x_1, x_2).
\]

(3.5)
We remark that since $z + \beta = z - \beta = u_1 + u_2 |\beta = u_1^0 |\beta = u_2^0 |\beta = 0$, then
\[ \mathbf{u}_1 |\beta = \mathbf{u}_2 |\beta = 0. \]

Since $\text{div} \mathbf{u} = 0$ in $\Omega$ we can write
\[ \partial_{x_3} u_3 = - (\partial_{x_1} u_1 + \partial_{x_2} u_2), \tag{3.6} \]
and then
\[ \int_{z_-}^{z_+} \partial_{x_3} u_3(x) dx = - \left( \int_{z_-}^{z_+} \partial_{x_1} u_1(x) dx + \int_{z_-}^{z_+} \partial_{x_2} u_2(x) dx \right). \]
holds for all $(x_1, x_2) \in \Omega$. By using the rule of the differentiation under integral sign we have for $i = 1, 2, 3$ that
\[ \partial_{x_i} \int_{z_-}^{z_+} u_i(x) dx = \int_{z_-}^{z_+} \partial_{x_i} u_i(x) dx = u_i(x_1, x_2, z_+ - u_i(x_1, x_2, z_-) \partial_{x_i} z_+ + \int_{z_-}^{z_+} \partial_{x_i} u_i(x) dx \]
\[ = u_i^e \partial_{x_i} z_+ - u_i^s \partial_{x_i} z_- + \int_{z_-}^{z_+} \partial_{x_i} u_i(x) dx. \]
Moreover $u_i^e \partial_{x_i} z_+ = u_i^s \partial_{x_i} z_- = 0$ in $\Omega$, because $u_i^e \neq 0$ if and only if $(x_1, x_2) \in \Omega^c$, however $\partial_{x_i} z_+ = h^{-1} \partial_{x_i} h_+ = 0$ for $(x_1, x_2) \in \Omega^c$, a similar situation occurs with $u_i^s$ and $\partial_{x_i} z_+ = 0$. In particular,
\[ \int_{z_-}^{z_+} \partial_{x_i} u_i(x) dx = \partial_{x_i} \left( (z_+ - z_-) \mathbf{u}_1(x_1, x_2) \right). \tag{3.7} \]
holds for $i = 1, 2$. Moreover, we have
\[ \int_{z_-}^{z_+} \partial_{x_3} u_3(x) dx = u_3^e - u_3^s \]
\[ = - \left( \int_{z_-}^{z_+} \partial_{x_1} u_1(x) dx + \int_{z_-}^{z_+} \partial_{x_2} u_2(x) dx \right) \]
\[ = - (\partial_{x_1} \left( (z_+ - z_-) \mathbf{u}_1(x_1, x_2) \right) + \partial_{x_2} \left( (z_+ - z_-) \mathbf{u}_2(x_1, x_2) \right)), \]
that is, (3.1) holds for all $(x_1, x_2) \in \Omega$. This proves the first statement of the theorem.

To show the second one, we use (3.2) to obtain
\[ \partial_{x_1} u_i = \partial_{x_1} \left( \frac{\sigma}{2} \partial_{x_2} q \right) (x_3 - z_+) (x_3 - z_-) - \frac{\sigma}{2} \partial_{x_1} q \partial_{x_2} z_+ \cdot (x_3 - z_-) - \frac{\sigma}{2} \partial_{x_1} q \partial_{x_2} z_- \cdot (x_3 - z_+) \]
\[ + \partial_{x_2} \left( \frac{u_i^e - u_i^s}{z_+ - z_-} \right) x_3 + \partial_{x_1} \left( \frac{u_i^e z_+ - u_i^s z_-}{z_+ - z_-} \right), \]
for $i = 1, 2$ and by using (3.6) we have
\[ u_3(x_1, x_2, x_3) = - (\partial_{x_1} \left( \frac{\sigma}{2} \cdot \partial_{x_1} q \right) + \partial_{x_2} \left( \frac{\sigma}{2} \cdot \partial_{x_2} q \right)) \left[ \frac{1}{2} (x_3 - z_+)^2 (x_3 - z_-) - \frac{1}{6} (x_3 - z_+)^3 \right] \]
\[ + \frac{\sigma}{2} \cdot (\partial_{x_1} q \partial_{x_1} z_+ + \partial_{x_2} q \partial_{x_2} z_-) \left[ \frac{1}{2} (x_3 - z_+)^2 \right] \]
\[ + \frac{\sigma}{2} \cdot (\partial_{x_1} q \partial_{x_1} z_+ + \partial_{x_2} q \partial_{x_2} z_-) \left[ \frac{1}{2} (x_3 - z_-)^2 \right] \]
\[ \left[ \frac{x_3^2}{2} \partial_{x_1} \left( \frac{u_i^e - u_i^s}{z_+ - z_-} \right) x_3 \partial_{x_1} \left( \frac{u_i^e z_+ - u_i^s z_-}{z_+ - z_-} \right) \right] \]
\[ = \left[ \frac{x_3^2}{2} \partial_{x_1} \left( \frac{u_i^e - u_i^s}{z_+ - z_-} \right) + x_3 \partial_{x_1} \left( \frac{u_i^e z_+ - u_i^s z_-}{z_+ - z_-} \right) - \kappa(x_1, x_2), \right. \tag{3.8} \]
for some function $\kappa : \Omega \to \mathbb{R}$. Since we known that
\[
u_2(x_1, x_2, z_+(x_1, x_2)) = u_3^s(x_1, x_2) \text{ for } (x_1, x_2) \in \Omega,
\]
then
\[
\kappa(x_1, x_2) = \frac{\sigma}{2} \cdot (\partial_{x_1} q \partial_{z_+} z_+ + \partial_{x_2} q \partial_{z_+} z_+) \left[ \frac{1}{2} (z_+ - z_-) \right] \\
- \frac{1}{2} \left[ \partial_{x_1} \left( \frac{u_1 - u_1}{z_+ - z_-} \right) + \partial_{x_2} \left( \frac{u_2 - u_2}{z_+ - z_-} \right) \right] z_+ - \frac{1}{2} \left[ \partial_{x_1} \left( \frac{u_1^2 z_+ - u_1^2 z_-}{z_+ - z_-} \right) + \partial_{x_2} \left( \frac{u_2^2 z_+ - u_2^2 z_-}{z_+ - z_-} \right) \right] z_+ - u_3^s.
\]
In a similar way, since
\[
u_3(x_1, x_2, z_-(x_1, x_2)) = u_3^s(x_1, x_2) \text{ for } (x_1, x_2) \in \Omega,
\]
also
\[
\kappa(x_1, x_2) = \frac{\sigma}{2} \cdot (\partial_{x_1} q \partial_{z_-} z_- + \partial_{x_2} q \partial_{z_-} z_-) \left[ \frac{1}{2} (z_- - z_+) \right] \\
- \frac{1}{2} \left[ \partial_{x_1} \left( \frac{u_1 - u_1}{z_+ - z_-} \right) + \partial_{x_2} \left( \frac{u_2 - u_2}{z_+ - z_-} \right) \right] z_- - \frac{1}{2} \left[ \partial_{x_1} \left( \frac{u_1^2 z_+ - u_1^2 z_-}{z_+ - z_-} \right) + \partial_{x_2} \left( \frac{u_2^2 z_+ - u_2^2 z_-}{z_+ - z_-} \right) \right] z_- - u_3^s,
\]
holds and the theorem follows.

### 3.1 A Darcy like equation

A first consequence of the above result is the following. From Theorem 3.1 we can write
\[
\begin{pmatrix}
\nabla_1(x_1, x_2) \\
\nabla_2(x_1, x_2)
\end{pmatrix} = - \left( \begin{array}{c}
\frac{1}{2} (z_+ - z_-) \sigma \\
\frac{1}{2} (z_+ - z_-) \sigma
\end{array} \right) \begin{pmatrix}
\partial_{x_1} q \\
\partial_{x_2} q
\end{pmatrix} - \frac{1}{2} \left( \begin{array}{c}
u_1^s \\
\nu_2^s
\end{array} \right),
\]
that is,
\[
\mathbf{v}(x_1, x_2) := \begin{pmatrix}
\nabla_1(x_1, x_2) \\
\nabla_2(x_1, x_2)
\end{pmatrix} = - \begin{pmatrix}
\partial_{x_1} q \\
\partial_{x_2} q
\end{pmatrix} = - \nabla q,
\]
where $\mathbf{v}(x_1, x_2)$ is a "velocity" defined by
\[
\begin{pmatrix}
\nabla_1(x_1, x_2) \\
\nabla_2(x_1, x_2)
\end{pmatrix} := \begin{pmatrix}
\frac{1}{2} (z_+ - z_-) \sigma \\
\frac{1}{2} (z_+ - z_-) \sigma
\end{pmatrix}^{-1} \begin{pmatrix}
\nabla_1(x_1, x_2) \\
\nabla_2(x_1, x_2)
\end{pmatrix} + \frac{1}{2} \left( \begin{array}{c}
u_1^s \\
\nu_2^s
\end{array} \right),
\]
such that $\mathbf{v}|_{\partial \Omega} = 0$ and
\[
\mathbf{v}(x_1, x_2) = \begin{pmatrix}
\frac{1}{2} (z_+ - z_-) \sigma \\
\frac{1}{2} (z_+ - z_-) \sigma
\end{pmatrix}^{-1} \begin{pmatrix}
\nabla_1(x_1, x_2) \\
\nabla_2(x_1, x_2)
\end{pmatrix} = - \nabla q,
\]
for $(x_1, x_2) \in \Omega \setminus (\Omega^s \cup \Omega^e)$. Moreover, (3.1) implies that the "pressure" $q = q(x_1, x_2)$ satisfies in $\Omega$ the Darcy like elliptic PDE:
\[
- \nabla \cdot \left( \begin{pmatrix}
\frac{1}{2} (z_+ - z_-) \sigma \\
\frac{1}{2} (z_+ - z_-) \sigma
\end{pmatrix} \nabla q \right) = \frac{1}{2} \nabla \cdot \left( \begin{pmatrix}
\nu_1^s \\
\nu_2^s
\end{pmatrix} \right) + u_3^s - u_3^e. \tag{3.9}
\]
It seems clear that if we solve equation (3.9) obtaining $q$, then by using (3.2) and (3.3) in Theorem 3.1 we can compute the velocity $\mathbf{u}$. This fact allows us to consider this 2.5D model as an approximation of (2.3)-(2.4) under the constraint of a two dimensional viscosity term. As usual the pressure gradient $\nabla q$ in the normal direction to the mold walls is zero, that is,
\[
\nabla q(x_1, x_2) \cdot \mathbf{n}(x_1, x_2) = 0 \text{ for } (x_1, x_2) \in \partial \Omega,
\]
Here \( n(x_1, x_2) \) denotes a normal vector to the boundary of \( \Omega \). Physically this means that material cannot leaves the mold cavity through the mold walls. This condition allows to an homogeneous neumann boundary condition, \( \nabla q(x_1, x_2) \cdot n(x_1, x_2) = g(x_1, x_2) = 0 \). It is well-known [7], that to ensure the existence of a solution the data

\[
f = \frac{1}{2} \nabla \cdot \left( \begin{pmatrix} u_1^e \\ u_2^e \end{pmatrix} + \begin{pmatrix} u_1^s \\ u_2^s \end{pmatrix} \right) + u_3^e - u_3^s
\]

and \( g = 0 \) must satisfy the compatibility relation

\[
\int_{\Omega} f + \int_{\partial\Omega} g = 0,
\]

that is,

\[
\int_{\Omega^*} \left( \frac{1}{2} \partial_{x_1} u_1^s + \frac{1}{2} \partial_{x_2} u_2^s + u_3^s \right) = \int_{\Omega^*} \left( u_3^s - \frac{1}{2} \partial_{x_1} u_1^e - \frac{1}{2} \partial_{x_2} u_2^e \right), \tag{3.11}
\]

must holds.

### 3.2 The Proper Generalized Decomposition

The Proper Generalized Decomposition (PGD) method has been recently proposed [3, 11, 13] for the a priori construction of separated representations of an element \( u \) in a tensor product space \( V = V_1 \otimes \cdots \otimes V_d \), which is the solution of a problem of linear functional equation. A rank-\( n \) approximated separated representation \( u_n \) of \( u \) is defined by

\[
u_n = \sum_{i=1}^{n} v_1^i \otimes \cdots \otimes v_d^i \tag{3.12}\]

The concept of separated representation was introduced by Beylkin and Mohlenkamp in [4] and it is related with the problem of constructing the approximate solutions of some classes of problems in high-dimensional spaces by means a separable function. In particular, for a given map

\[
u : [0, 1]^d \subset \mathbb{R}^d \rightarrow \mathbb{R},
\]

we say that it has a **separable representation** if

\[
u(x_1, \ldots, x_d) = \sum_{j=1}^{\infty} u_1^{(j)}(x_1) \cdots u_d^{(j)}(x_d) \tag{3.13}\]

Now, consider a mesh of \([0, 1]\) in the \(x_k\)-variable given by \( N_k \)-mesh points, \( 1 \leq k \leq d \), then we can write a discrete version of (3.13) by

\[
u(x_{i_1}, \ldots, x_{i_d}) = \sum_{j=1}^{\infty} u_1^{(j)}(x_{i_1}) \cdots u_d^{(j)}(x_{i_d}), \tag{3.14}\]

where \( 1 \leq i_k \leq N_k \) for \( 1 \leq k \leq d \). Observe that for each \( 1 \leq k \leq d \), if \( x_k^j \in \mathbb{R}^{N_k} \) denotes the vector with components \( u_k^{(j)}(x_{i_k}) \) for \( 1 \leq i_k \leq N_k \), then (3.14) is equivalent to

\[
u = \sum_{j=1}^{\infty} x_1^j \otimes \cdots \otimes x_d^j. \tag{3.15}\]

Our starting point is to consider that the reference set can be written as \( \Omega = I_1 \times I_2 \) where \( I_1 \) and \( I_2 \) are open intervals in the real line \( \mathbb{R} \) (see Example 2.1). We can assume that \( q \in H^1(\Omega) \) and we introduce its closed subspace (and also a Hilbert space)

\[
H^1_{f=0}(\Omega) = \{ \varphi \in H^s(\Omega) : \int_{\Omega} \varphi = 0 \}.
\]
We remark that Proposition 3.2 The sequence \( m \) for each \( q \) previous computed pressure \( f_i, f_2(x_2) : f_i \in H_{f=0}^s(I_i), i = 1, 2 \) 

Let us consider the set of tensors of bounded rank one in \( H_{f=0}^s(\Omega) \) given by 
\[
\mathcal{M}_1 \left( H_{f=0}^s(\Omega) \right) = \left\{ f_1(x_1) f_2(x_2) : f_i \in H_{f=0}^s(I_i), i = 1, 2 \right\}.
\]

From Proposition 4.19 in [8], it can be shown that this \( \mathcal{M}_1 \left( H_{f=0}^s(\Omega) \right) \) is weakly closed in \( H_{f=0}^s(\Omega) \).

Now, multiplying the PDE (3.9) by a (sufficiently smooth) test function \( \varphi \) vanishing at the boundary, integrating over \( \Omega \), and using the Green formula yields
\[
\int_{\Omega} \left( \frac{1}{2} \frac{(z_i - z_j)^3}{2} \cdot \sigma \right) \nabla q \cdot \nabla \varphi = \int_{\Omega} \left( \frac{1}{2} \frac{(z_i - z_j)^3}{2} \cdot \sigma \right) \nabla u_1 \cdot \nabla \varphi + \int_{\Omega} \left( \frac{1}{2} \frac{(z_i - z_j)^3}{2} \cdot \sigma \right) \nabla u_2 \cdot \nabla \varphi \quad (3.16)
\]

and where we assume that the data functions \( u_1 \) and \( u_2 \) for \( i = 1, 2, 3 \) appearing in the right term of (3.16), satisfy the compatibility condition (3.11). Now, take the functional \( J : H_{f=0}^1(\Omega) \rightarrow \mathbb{R} \) defined by
\[
J(q) := \frac{1}{2} \int_{\Omega} \left( \frac{1}{2} \frac{(z_i - z_j)^3}{2} \cdot \sigma \right) \nabla q \cdot \nabla q - \int_{\Omega} \left( \frac{1}{2} \frac{(z_i - z_j)^3}{2} \cdot \sigma \right) \nabla u_1 \cdot \nabla q + \int_{\Omega} \left( \frac{1}{2} \frac{(z_i - z_j)^3}{2} \cdot \sigma \right) \nabla u_2 \cdot \nabla q + u_3^s - u_3^s \cdot q.
\]

Then
\[
q \in \arg\min_{q \in H_{f=0}^1(\Omega)} J(q)
\]

if and only if (3.16) holds for all \( \varphi \in H_{f=0}^1(\Omega) \).

3.2.1 The PGD procedure

Now, consider the following procedure. We let \( q_0 = 0 \), and for \( m \geq 0 \) we construct \( q_m \in H_{f=0}^1(\Omega) \) from a previous computed pressure \( q_{m-1} \in H_{f=0}^1(\Omega) \) as follows: Find an element \( z_m \in \mathcal{M}_1 \left( H_{f=0}^s(\Omega) \right) \) such that
\[
z_m \in \arg\min_{z \in \mathcal{M}_1 \left( H_{f=0}^s(\Omega) \right)} J(q_{m-1} + z).
\]

We remark that
\[
q_m(x_1, x_2) = \sum_{j=1}^{m} \delta_1^{(j)}(x_1) \delta_2^{(j)}(x_2)
\]

for each \( m \geq 1 \). From Theorem 5 in [9] the next result follows.

**Proposition 3.2** The sequence \( \{q_m\}_{m \geq 0} \) converges to \( q \in \arg\min_{q \in H_{f=0}^1(\Omega)} J(q) \) strongly in \( H_{f=0}^s(\Omega) \).
4 A particular 2.5D model

In this section we particularize our model to a mould with a two dimensional quadrilateral reference domain. More precisely, given a fixed $\delta > 0$ let us consider the mould defined in Example 2.1. In this example we have

$$n_e = (0, 0, 1) \text{ and } n_s = (0, 0, -1),$$

and hence we can write the inlet and outlet as

$$v_e = (v^e_1, v^e_2, -v^e_3) \text{ and } v_s = (v^s_1, v^s_2, -v^s_3),$$

respectively, because

$$v_e \cdot n_e < 0 \text{ and } v_s \cdot n_s > 0.$$

Moreover, $h = 2\delta$. After the change of variables we have

$$W' = \{(x_1, x_2, x_3) : (x_1, x_2) \in \Omega \text{ and } z_+(x_1, x_2) < x_3 < z_-(x_1, x_2)\},$$

where $z_+, z_- : -\delta, 1 + \delta[|\cdot|] - \delta, 1 + \delta[\rightarrow \mathbb{R}]$ are defined by

$$z_+(x_1, x_2) := \frac{h^e(x_1, x_2)}{2\delta} = \begin{cases} \frac{1}{2} & \text{if } (x_1, x_2) \in [0, 1] \times [0, 1] \\ \frac{\delta + x_1}{2\delta} + \frac{1}{2} & \text{if } (x_1, x_2) \in \Delta_1, \\ \frac{1}{2} + \frac{1-x_1}{2\delta} & \text{if } (x_1, x_2) \in \Delta_2, \\ \frac{1}{2} + \frac{1-x_2}{2\delta} & \text{if } (x_1, x_2) \in \Delta_3, \\ \frac{\delta + x_2}{2\delta} + \frac{1}{2} & \text{if } (x_1, x_2) \in \Delta_4, \end{cases}$$

and

$$z_-(x_1, x_2) := \frac{h^s(x_1, x_2)}{2\delta} = -z_+(x_1, x_2).$$

Then

$$G^e_s := \{(x_1, x_2, 1/2) : 0 \leq x_1 \leq \delta \text{ and } 0 \leq x_2 \leq 1\}$$

and

$$(z_+ - z_-)(x_1, x_2) = \begin{cases} 1 & \text{if } (x_1, x_2) \in [0, 1] \times [0, 1] \\ \frac{\delta + x_1}{2\delta} + 1 & \text{if } (x_1, x_2) \in \Delta_1, \\ 1 + \frac{1-x_1}{\delta} & \text{if } (x_1, x_2) \in \Delta_2, \\ 1 + \frac{1-x_2}{\delta} & \text{if } (x_1, x_2) \in \Delta_3, \\ \frac{\delta + x_2}{2\delta} + 1 & \text{if } (x_1, x_2) \in \Delta_4. \end{cases}$$

Assume a physical model where the velocities at the inlet and the outlet are uniformly distributed, that is, at the same time every point in the inlet (respectively, outlet) have the same velocity. Under the above assumption, the velocities at the inlet and the outlet can be written as

$$u_e(t) = (u^e_1(t), u^e_2(t), u^e_3(t))\chi_{[0,1]}(x_1)\chi_{[0,1]}(x_2)$$

and

$$u_s(t) = (u^s_1(t), u^s_2(t), u^s_3(t))\chi_{[-1,1]}(x_1)\chi_{[0,1]}(x_2),$$

respectively. Then,

$$f = \frac{1}{2} \nabla \cdot \left( \begin{pmatrix} u^e_1 \\ u^e_2 \\ u^e_3 \\ u^s_1 \\ u^s_2 \end{pmatrix} \right) + u^e_3 - u^e_3 - u^s_3 - u^s_3,$$
and hence the compatibility condition (3.11) means
\[ u_3^t \delta = u_3^a \delta \text{ or } u_3^e = u_3^a, \]
for all time \( t \). Now, recall that the discharge is equal to the product of the stream’s cross-sectional area and its mean velocity. Since \( u_3^e \delta \) is the entry discharge and \( u_3^a \delta \) the exit discharge. The compatibility condition implies the total entry discharge must be equal to the total exit one. A variational formulation of (3.9) allows us to find \( q \in H^1_{f=0}(I) \) such that
\[
\int_{-\delta}^{1+\delta} \int_{-\delta}^{1+\delta} \frac{2 \sigma z^3}{3} \nabla q \cdot \nabla \varphi \, dx_1 \, dx_2 = - \int_{-\delta}^{1+\delta} \int_{-\delta}^{1+\delta} f \varphi \, dx_1 \, dx_2
\]
holds for all \( \varphi \in H^1_{f=0}(I) \), where the source term is
\[ f = u_3^e(t) \left( \chi_{[0,x]}(x_1) \chi_{[0,1]}(x_2) - \chi_{(-\delta,1]}(x_1) \chi_{[0,1]}(x_2) \right). \]
Observe that \( f \in H^1_{f=0}(I) \), can be written as
\[ f(x_1,x_2) = g_1(x_1) \chi_{[0,1]}(x_2), \]
where
\[ g_1(x_1) := u_3^e(t) \left( \chi_{[0,x]}(x_1) - \chi_{(-\delta,1]}(x_1) \right). \]
Thus, \( f \in \mathcal{M}_1(H^1_{f=0}(I)) \). Moreover, the variational formulation of (4.1) means to find \( q \in H^1_{f=0}(I) \) such that
\[
\int_{-\delta}^{1+\delta} \int_{-\delta}^{1+\delta} \frac{\sigma (z_+ - z_-)^3}{12} \nabla q \cdot \nabla (\varphi_1 \otimes \varphi_2) \, dx_1 \, dx_2 = \left( \int_{-\delta}^{1+\delta} g_1 \varphi_1^{(\alpha)} \, dx_1 \right) \left( \int_{-\delta}^{1+\delta} \varphi_2^{(\beta)} \, dx_2 \right)
\]
holds for all \( \varphi_1^{(\alpha)} \in H^1_{f=0}(I_1) \) and \( \varphi_2^{(\beta)} \in H^1_{f=0}(I_2) \).

Next, we give a procedure in order to perform a numerical simulation of a mould filling process. To this end, we recall that the function
\[ \sigma = \sigma(s(t,x_1,x_2),x_1,x_2), \]
depends on the pseudo-concentration function \( s \), which satisfy \( \partial_{x_i}s = 0 \), and hence
\[
\partial_t s + \sum_{i=1}^2 u_i \partial_{x_i}s = 0 \text{ in } (0,T) \times W', \tag{4.3}
\]
\[ s(t,x) = 1 \text{ on } (0,T) \times G'_{t}, \tag{4.4}
\]
\[ s(0,x) = 0 \text{ for all } x \in W'. \tag{4.5}
\]
Since \( s(t=0,x_1,x_2) \) is well-known and then we can compute the pressure \( q(t=0,x_1,x_2) \) by using (4.1), and, from (3.2) and (3.3), we obtain \( u(t=0,x) \) for all \( x \in W' \). Recall that
\[ u(t,x) = 0 \text{ in } \partial W' \setminus (G'_{t} \cup G'_{s}) \]
\[ u(t,x) = u_0(t,x) \text{ in } G'_{t}, \]
\[ u(t,x) = u_0(t,x) \text{ in } G'_{s}, \]
for all time \( t \). Since we known the velocity field for all time \( t \) in \( G'_{t} \) and \( G'_{s} \) and \( u(0,x) \) for all \( x \in W' \), we can extend the velocity field for all time \( t \) over the sets
\[ G_{t}^{(0)} := \{(x_1,x_2,x_3) \in W' : 0 \leq x_1 \leq \delta, 0 \leq x_2 \leq 1 \text{ and } z_-(x_1,x_2) < x_3 < z_+(x_1,x_2) \} \]
and
\[ G_{s}^{(0)} := \{(x_1,x_2,x_3) \in W' : 1 - \delta \leq x_1 \leq 1, 0 \leq x_2 \leq 1 \text{ and } z_-(x_1,x_2) < x_3 < z_+(x_1,x_2) \}, \]
by interpolation, as follows. We put for \( s \in [t, t + \Delta t) \)

\[
\mathbf{u}(s, x) = \Phi_s(s, \mathbf{u}(t + \Delta t, x) \subset G_t^{(0)}),
\]

\[
\mathbf{u}(s, x) = \Phi_s(s, \mathbf{u}(t, x) \subset G_t^{(0)}),
\]

for some interpolating functions \( \Phi_s \) and \( \Phi_s \) satisfying

\[
\Phi_s(t, \mathbf{u}(t, x) = \mathbf{u}(t, x),
\]

\[
\Phi_s(t, \mathbf{u}(t, x) = \mathbf{u}(t, x),
\]

\[
\Phi_s(t, \mathbf{u}(t, x) = \mathbf{u}(t, x),
\]

\[
\Phi_s(t, \mathbf{u}(t, x) = \mathbf{u}(t, x),
\]

Observe that \( G_t^{(0)} = \partial G_t^{(0)} \) and \( G_t^{(0)} = \partial G_t^{(0)} \). Our next step is to construct a finite sequence of functions \( \{\mathbf{u}^{(j)}(t, x)\}_{j=1}^{m} \), for \( 0 \leq t \leq j\Delta t \) and \( x \in W' \), such that \( \mathbf{u}^{(j)}(t, x) = \mathbf{u}(t, x) \) holds for all \( x \in G_t^{(0)} \cup G_t^{(0)} \) and \( 0 \leq t \leq j\Delta t \).

In order to construct \( \mathbf{u}^{(1)}(t, x) \), for \( t \in [0, \Delta t] \) and \( x \in W' \), we may assume that \( \mathbf{u}^{(1)}(t, x) = \mathbf{u}(0, x) \) in \( W' \setminus G_t^{(1)} \cup G_t^{(1)} \) where

\[
G_t^{(1)} := \{(x_1, x_2, x_3) \in W' : \delta \leq x_1 \leq \delta + \Delta x_1, 0 \leq x_2 \leq 1 \text{ and } z_-(x_1, x_2) < x_3 < z_+(x_1, x_2)\}
\]

\[
G_t^{(1)} := \{(x_1, x_2, x_3) \in W' : 1 - \delta - \Delta x_1 \leq x_1 \leq 1 - \delta, 0 \leq x_2 \leq 1 \text{ and } z_-(x_1, x_2) < x_3 < z_+(x_1, x_2)\}
\]

and we put

\[
\mathbf{u}^{(1)}(t, x) = \Pi_s(t, \mathbf{u}(t, x) \subset G_t^{(1)}),
\]

\[
\mathbf{u}^{(1)}(t, x) = \Pi_s(t, \mathbf{u}(t, x) \subset G_t^{(1)}),
\]

for some interpolating functions \( \Pi_s, \Pi_s \) satisfying

\[
\Pi_s(0, \mathbf{u}(t, x) = \mathbf{u}(0, x),
\]

\[
\Pi_s(0, \mathbf{u}(t, x) = \mathbf{u}(0, x),
\]

\[
\Pi_s(0, \mathbf{u}(t, x) = \mathbf{u}(0, x),
\]

\[
\Pi_s(0, \mathbf{u}(t, x) = \mathbf{u}(0, x),
\]

Now, we put \( \mathbf{u}(t, x) = \mathbf{u}^{(1)}(t, x) \) for \( t \in [0, \Delta t] \), and then we can compute \( s(t, x) \) for \( t \in [0, \Delta t] \) from

\[
\frac{\partial s}{\partial t} + \sum_{i=1}^{3} u_i \frac{\partial s}{\partial x_i} = 0 \text{ in } (0, \Delta t) \times W',
\]

\[
s(t, x) = 1 \text{ on } (0, \Delta t) \times G_t^{(1)},
\]

\[
s(0, x) = 0 \text{ for all } x \in W'.
\]

From the knowledge of \( s(t = \Delta t, x) \) we compute \( q(t = \Delta t, x) \) by using (4.1), and, from (3.2) and (3.3), we obtain \( \mathbf{u}(t = \Delta t, x) \) for all \( x \in W' \). Write \( \mathbf{u}^{(1)}(t = \Delta t, x) = \mathbf{u}(t = \Delta t, x) \) and then \( \mathbf{u}^{(1)}(t = \Delta t, x) \) is the approximating velocity field for \( 0 \leq t \leq \Delta t \) and \( x \in W' \).

We take

\[
\Delta x_1 = \frac{1 - 2\delta}{2m} \text{ and } \Delta t = \frac{T}{m},
\]

for some natural number \( m > 1 \), and hence \( \delta + m\Delta x_1 = 1 - (\delta + m\Delta x_1) \) holds for such \( m \).

For a fixed \( j \geq 2 \), we have the knowledge of \( \mathbf{u}^{(j-1)}(t, x) \) for \( (t, x) \in [0, (j-1)\Delta t] \) \times W'. Then for \( 0 \leq t < j\Delta t \) we assume that \( \mathbf{u}^{(j)}(t, x) = \mathbf{u}^{(j-1)}(t, x) \) for all \( x \in W' \setminus (G_t^{(1)} \cup G_t^{(1)}) \), where

\[
G_t^{(j)} := \{(x_1, x_2, x_3) \in W' : \delta \leq x_1 \leq \delta + j\Delta x_1, 0 \leq x_2 \leq 1 \text{ and } z_-(x_1, x_2) < x_3 < z_+(x_1, x_2)\}
\]

\[
G_t^{(j)} := \{(x_1, x_2, x_3) \in W' : 1 - \delta - j\Delta x_1 \leq x_1 \leq 1 - \delta, 0 \leq x_2 \leq 1 \text{ and } z_-(x_1, x_2) < x_3 < z_+(x_1, x_2)\}.
\]
and we put
\[ u^{(j)}(t, x_1, x_2, x_3) = \Pi_e(t, u(j\Delta t, \delta, x_2, x_3), u^{(j-1)}(t, x_1, x_2, x_3)) \quad \text{for} \quad (x_1, x_2, x_3) \in G_e^{(j)}, \]
\[ u^{(j)}(t, x_1, x_2, x_3) = \Pi_s(t, u(j\Delta t, 1-\delta, x_2, x_3), u^{(j-1)}(t, x_1, x_2, x_3)) \quad \text{for} \quad (x_1, x_2, x_3) \in G_s^{(j)}, \]
for some interpolating functions also denoted by \( \Pi_e, \Pi_s \), satisfying
\[ \Pi_e(0, u(j\Delta t, \delta, x_2, x_3), u^{(j-1)}(0, x_1, x_2, x_3)) = u^{(j-1)}(0, x_1, x_2, x_3), \]
\[ \Pi_e(j\Delta t, u(j\Delta t, \delta, x_2, x_3), u^{(j-1)}(j\Delta t, \delta, x_2, x_3)) = u(j\Delta t, \delta, x_2, x_3), \]
\[ \Pi_s(0, u(j\Delta t, 1-\delta, x_2, x_3), u^{(j-1)}(0, x_1, x_2, x_3)) = u^{(j-1)}(0, x_1, x_2, x_3), \]
\[ \Pi_s(j\Delta t, u(j\Delta t, 1-\delta, x_2, x_3), u^{(j-1)}(j\Delta t, 1-\delta, x_2, x_3)) = u(j\Delta t, 1-\delta, x_2, x_3). \]

Now, we put \( u(t, x) = u^{(j)}(t, x) \) for \( t \in [0, j\Delta t] \), and then we can compute \( s(t, x_2) \) for \( t \in [0, j\Delta t] \) from
\[
\partial_t s + \sum_{i=1}^{3} u_i \partial_{x_i}s = 0 \quad \text{in} \quad (0, j\Delta t) \times W',
\]
\[
s(t, x) = 1 \quad \text{on} \quad (0, j\Delta t) \times G_e',
\]
\[
s(0, x) = 0 \quad \text{for all} \quad x \in W'.
\]

From the knowledge of \( s(t = j\Delta t, x_1, x_2) \) we compute \( q(t = j\Delta t, x_1, x_2) \) by using \((4.1)\), and, from \((3.2)\) and \((3.3)\), we obtain \( u(t = j\Delta t, x) \) for all \( x \in W' \).

We remark that after \( m \)-steps we have \( u^{(m)}(t, x) \) for all \( (t, x) \in [0, T] \times W' \).

References


