A Direct LU Solver for Pricing American Bond Options under Hull-White model

A. Falcó\textsuperscript{a}, Ll. Navarro\textsuperscript{b,1}, C. Vázquez\textsuperscript{c}

\textsuperscript{a}Dpto. de Ciencias Físicas, Matemáticas y Computación, Univ. CEU Cardenal Herrera, Alfara del Patriarca, 46115, Valencia, Spain
\textsuperscript{b}Dpto. de Valoraciones, CRF, LLC, Castellana 93, 28043, Madrid, Spain
\textsuperscript{c}Dpto. de Matemáticas, Univ. da Coruña, Campus de Elviña, 15071, A Coruña, Spain

Abstract

The main goal of this paper is to propose a novel numerical algorithm to price American options on bonds. For this purpose, we illustrate the performance of this method by means of the valuation of an American Put Option on a discount bond under the extended Vasicek model due to Hull and White (HW) and using the consistent forward rate curves. In particular, an implicit Crank-Nicolson (CN) scheme in time is applied obtaining a discretized linear complementarity problem (LCP) and then we introduce a direct LU based method to solve the LCP. Finally, we carry out numerical experiments to examine the convergence of this method and to testify the efficiency and effectiveness of this numerical scheme against other standard approaches.

Keywords: American Bond Options, Interest Rate Models, Crank-Nicolson Method, Linear Complementarity Problem, LU decomposition

1. Introduction

The present work analyzes in depth the valuation of American options on zero-coupon bond using the one-factor Hull-White model (see [10, pp. 577–581] and [4, Chap. 3, pp. 73–80] for an excellent survey).

Market models have recently emerged as a market standard for the pricing of exotic interest rate products. In these models the dynamics of Libor and swap rates are directly

Email addresses: afalco@uchceu.es (A. Falcó), lluis.navarro@crfsa.es (Ll. Navarro), carlosv@udc.es (C. Vázquez)

\textsuperscript{1}The author acknowledge financial support from Generalitat Valenciana grant PROMETEOII/2013/015.

Preprint submitted to Elsevier May 14, 2016
specified. In spite of their increasing popularity, market models have one serious drawback. An accurate implementation can only be made through simulation, which is typically slow due to the large number discrete-tenored market rates needed to be evolved through time. This problem may be acute for products with an early exercise provision, such as American-style bond options, as simulation is not naturally suited for performing backward-in-time calculations needed to determine the optimal exercise strategy.

In contrast to market models, the Hull-White model incorporates a few appealing properties. First, it is analytically tractable.

Second, path-independent products with an early exercise provision can also be efficiently evaluated by using finite differences methods.

The computational efficiency offered by the Hull-White model explains why many banks still use this kind of model for mark to market, risk calculation and other management purposes even after the introduction of the market models. Despite its popularity, rigorous studies which compare different alternatives of its numerical implementation by means of pure deterministic methods, such as the finite difference methods, are scarce.

Vetzal [18] made a case that using implicit methods could improve upon the performance of the more traditionally used trinomial-tree based techniques. While this is handy, it is unclear how Vetzal’s implicit finite differences approach actually considers boundary conditions and to what extent this choice may affect the convergence to the correct solution in some specific valuation problems. In fact, Vetzal’s alternative discretization strategy, is essentially the same as the utilised in [11, 17] for numerical implementations of the explicit-type in order to avert the requirement of the specification of spatial boundary conditions. However, it can be proved for all three approaches that is indeed equivalent to impose the discretized version of the PDE in the boundary, a technical point that is difficult to explain from PDE literature perspective and even worst, rather nonsensical from a theoretical point of view.

More recently, in [6] a Crank-Nicolson method for valuing interest rate derivatives is proposed for some specific consistent models.

Although the method extends the seminal methodology [11, 12] of Hull and White allowing the use of implicit methods, again the apparent lack of boundary conditions is definitely a sign of an heterodox use of the discretized version of each PDE at the boundary. Moreover, in order to deal with the Kolmogorov Forward Equation (henceforth KFE), the forward induction approach introduced by Jamshidian [15] is adapted to be suitable for a pure implicit
scheme, unfortunately this feature adds computational costs to the whole numerical method. Last but not least, none of these works have the solution to the early-exercise problem when implicit methods are taking into account.

Due to the complexity of free-boundary problems, we rely on numerical experiments. We first reduce the Hull-White PDE after some transformations with the help of a reference variable. Second, as finite difference methods are straightforward to implement we compare explicit and Crank-Nicolson methods with the more traditional lattice-based approaches for European options. Finally, we consider American-style bond options and the linear complementarity problem (LCP) related with their valuation. For the Crank-Nicolson implementations, we solve the LCP using LU decomposition and a modified backward substitution with a projection. Experiments show that the direct LU algorithm is much faster and robust than the projected SOR method. The overall numerical results show that a Crank-Nicolson implementation is efficient and robust outperforming the rest.

The outline of the work is as follows. In Section 1 we briefly review the Hull-White model.

Next, in Section 2, we transform the Hull-White PDE into a more simple diffusion equation. In Section 3, we describe the application of the Finite Difference $$\theta$$-scheme to this reduced PDE. In Section 4, the numerical results, for both explicit and Crank-Nicolson methods, are presented and compared with the closed forms solutions (European-style options) and those obtained with the lattice method developed in [11]. A direct LU algorithm for the complementarity problem resulting from pricing American-style options and linked numerical experiments are the subject of Section 5. Finally, Section 6 contains conclusions.

2. The Model

We recall the risk neutral dynamics of the Hull-White model (henceforth HW):

$$dr = [\Phi(t) - ar] dt + \sigma dW^Q.$$  \hspace{1cm} (1)

where $$r$$ denotes the short rates evolution process, $$a$$ and $$\sigma$$ are the model parameters and $$W$$ denotes a Wiener process under the risk neutral measure $$Q$$.

It is well known that the price $$P(r, t, S)$$ at time $$t$$ of a pure discount bond with face value 1 monetary unit at its maturity date $$S$$ is given as follows

$$P(r, t, S) = A(t, S)e^{-B(t, S)r},$$  \hspace{1cm} (2)
where
\[ B(t, S) = \frac{1 - e^{-a(S-t)}}{a} , \]  
(3)

and
\[ \log A(t, S) = \log \frac{P(0,S)}{P(0,t)} - B(t, S)\partial_t \log P(0,t) - \frac{1}{4a^3} \sigma^2 (e^{-aS} - e^{-at})^2 (e^{2at} - 1) . \]  
(4)

Let \( V = V(r,t) \) be the value of a contingent claim on a zero-coupon bond where the holder can receive a given payoff \( g(r,t) \) at the expiry date \( t = T \). The option pricing problem can be formulated as the following PDE
\[
\begin{align*}
\frac{\sigma^2}{2} \partial_{rr} V + (\Phi(t) - ar) \partial_r V - rV + \partial_t V &= 0 , \quad (r, t) \in \mathbb{R} \times ]0, T[, \\
V(r,T) &= g(r,T) , \\
& \quad r \in \mathbb{R}.
\end{align*}
\]  
(5)

For instance, for the discount bond value we have
\[ g(r, T) = 1, \]
and for the European-style option on a bond the corresponding payoff is
\[ g(r, T) = [\phi(P(r, T, S) - K)]^+, \]
where \( T \) denotes the option expiry, which is prior to bond maturity \( S \) (i.e. \( T < S \)), \( P \) is the price of the underlying bond, \( K \) is the strike of the option and the binary unit \( \phi = +1 \) for the call and \( \phi = -1 \) for the put.

For a vanilla American-style put on a discount bond, the pricing requires to solve a linear complementarity problem associated to the PDE (5), which can be formulated as the following partial differential complementarity problem (PDCP).
\[
\begin{align*}
\mathcal{A}_{r,t} V(r,t) &\leq 0 , \\
V(r,t) - g(r,t) &\geq 0 , \\
(\mathcal{A}_{r,t} V(r,t)) (V(r,t) - g(r,t)) &= 0.
\end{align*}
\]  
(6)
a.e. in \( \mathbb{R} \times ]0, T[ \), where
\[ \mathcal{A}_{r,t} := \frac{\sigma^2}{2} \partial_{rr} + (\Phi(t) - ar) \partial_r - r + \partial_t \]  
(7)

\( ^2 \partial_x = \frac{\partial}{\partial x} \) is the partial derivative with respect to \( x \).
is a linear partial differential operator. The final condition is
\[ V(r, T) = g(r, T) = (K - P(r, T, S))^+, \quad r \in \mathbb{R}, \quad \text{for a put} \]
and the boundary conditions (behaviour at infinity) are
\[
\begin{align*}
V(r, t) &= g(r, t), \quad r \to \infty, \\
V(r, t) &= g(r, t), \quad r \to -\infty.
\end{align*}
\]

It is a well documented feature that the HW model may be translated into a one-factor HJM formulation \[9\], with a deterministic and quasi-exponential specification of the forward volatility
\[ \sigma_F(t, T) = \sigma e^{-a(T-t)}. \]
Therefore, starting from (1), the resulting short-rate dynamics is given by
\[
dr = \left[ \partial_t F(0, t) + a F(0, t) + \frac{\sigma^2}{2a} (1 - e^{-2at}) - ar \right] dt + \sigma dW, \tag{8}
\]
which is equivalent to (1) when combined with the identity
\[
\Phi(t) := \partial_t F(0, t) + a F(0, t) + \frac{\sigma^2}{2a} (1 - e^{-2at}). \tag{9}
\]
Furthermore, it is well-known that this model has the minimal consistent family
\[
F(0, t) := G_{MC}(z, a ; t) = z_1 e^{-at} + z_2 e^{-2at}, \quad 0 \leq t \leq T, \tag{10}
\]
as primarily shown in [1] by Björk and Gombani (see [2] and references therein for a detailed description of this kind of interpolant families).

We remark that (9) allows the calibration (fit) of this model to the initial yield curve. Finally, the main and celebrated advantage of HW model is the existence of exact solutions for the governing equations for discount bond prices and for the plain vanilla European options written on them (see the original derivation of these closed-form expressions in [10, pp. 571–579] and consult [8, pp. 109-111] for a modern derivation based on the HJM formulation). This benefit not only enables the numerical results to be benchmarked with the closed-form solutions at least for the more plain vanilla derivatives, but also makes possible to speed up the convergence of finite-difference methods by using exact prices on the mesh and their asymptotic behaviour for the set of boundary points in the bounded computational domain. 

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2.1. A Reduced Mathematical Problem

Now we describe in detail how the HW model we are considering can be fitted to the initial term structure of interest rates analytically, which is definitively different from both the original algorithm of Hull and White [11] and the implicit algorithm presented in [6] where the use of the forward induction technique [15] stands. Later, numerical examples confirm that this approach improves computation times and accuracy for option pricing whatever method of discretization is used, either explicit or implicit.

First, we consider the following transformation of variables

\[
x = r - \Omega(t) \\
u = e^{\int_{0}^{T} \Omega(q) dq} V,
\]

where

\[
\Omega(t) := e^{-at} \left( r_0 + \int_{0}^{t} e^{au} \Phi(u) du \right).
\]

Let us note that \(\Omega(t)\) is chosen in such a way that \(x_0 = 0\). The price of any derivative in terms of the new variable is denoted by \(w(x, t)\). We can immediately infer the following relationships

\[
V(r, t) = w(x, t) := e^{\int_{0}^{T} \Omega(q) dq} u(x, t) \\
\partial_t V = \partial_t w - (-a\Omega(t) + \Phi(t)) \partial_x w \\
\partial_r V = \partial_x w \\
\partial_{rr} V = \partial_{xx} w,
\]

where we have used that

\[
\frac{d}{dt} \Omega(t) + a\Omega(t) = \Phi(t).
\]

Note that the function \(\Omega(t)\) may be easily obtained by direct integration of (11):

\[
\Omega(t) = F(0, t) + \frac{\sigma^2}{2a^2} \left( 1 - e^{-at} \right)^2.
\]

Next, by using the identities from (12) and that \(r = x + \Omega(t)\) we obtain

\[
\mathcal{A}_x V(r, t) = \mathcal{A}_x w(x, t) = e^{\int_{0}^{T} \Omega(q) dq} \mathcal{B}_x u(x, t),
\]

where

\[
\begin{aligned}
\mathcal{A}_x w &= \frac{\sigma^2}{2} \partial_{xx} w - a x \partial_x w + (x + \Omega(t)) w + \partial_t w & \text{for} & \ (x, t) \in \mathbb{R} \times ]0, T[, \\
\mathcal{B}_x u &= \frac{\sigma^2}{2} \partial_{xx} u - a x \partial_x u - x u + \partial_t u & \text{for} & \ (x, t) \in \mathbb{R} \times ]0, T[ \text{ and} \\
w(x, T) = u(x, T) = h(x + \Omega(T), T) & , & x \in \mathbb{R}.
\end{aligned}
\]
Notice that the coefficients of the operator \( B_{x,t} \) no longer depend on \( t \). Now, the problem (6) can be rewritten in the form

\[
\begin{cases}
A_{x,t}w(x,t) \leq 0, \\
w(x,t) - g(x + \Omega(t), t) \geq 0, \\
(A_{x,t}w(x,t))(w(x,t) - g(x + \Omega(t), t)) = 0.
\end{cases}
\] (16)

a.e. in \( \mathbb{R} \times ]0, T[ \), so that the final condition is

\[
w(x, T) = g(x + \Omega(T), T) = (K - P(x + \Omega(T), T, S))^+, \quad r \in \mathbb{R}, \quad \text{for a put},
\]

and the boundary conditions are

\[
w(x, t) = g(x + \Omega(t), t), \quad x \to \infty, \\
w(x, t) = g(x + \Omega(t), t), \quad x \to -\infty.
\]

In this setting, by using (13) we can compute

\[
\int_T^t \Omega(q) dq = - \log \left( \frac{P(0, T)}{P(0, t)} \right) + \\
\quad + \frac{\sigma^2}{2a^2} \left( a(T - t) - 2(e^{-at} - e^{-aT}) + \frac{1}{2}(e^{-2at} - e^{-2aT}) \right). \tag{17}
\]

We also note that the term

\[- (\log P(0, T) - \log P(0, t))
\]

may be inferred with enough discount bond curve data

\[
\log P^o = \begin{bmatrix} \log P^o(0, x^1) & \log P^o(0, x^2) & \ldots & \log P^o(0, x^P) \end{bmatrix},
\]

and, for instance, the use of the minimal consistent family to fit the model in the continuous finite region \( I = [0, T] \)

\[
\log P(0, t) = \frac{z_1}{a} \left( e^{-at} - 1 \right) + \frac{z_2}{2a} \left( e^{-2at} - 1 \right), \quad 0 \leq t \leq T \leq x^P. \tag{18}
\]

3. A discrete approach to the partial differential complementarity problem

In this section we provide a discrete approach to (16) in order to give a numerical approximation to the price of an American put option on a bond. To this end let us consider a general form of a partial differential linear operator \( \mathcal{G} \) given by:

\[
\mathcal{G}_{x,t}z(x,t) := \partial_t z + a(x,t)\partial_{xx}z + b(x,t)\partial_xz + c(x,t)z,
\] (19)
defined over a suitable class of functions \( z(x, t) \). Now, consider the LCP problem associated to a PDE to find a function \( z \), such that

\[
\begin{align*}
G(x, t)z(x, t) &\leq 0, \\
z(x, t) - g(x, t) &\geq 0, \\
(G(x, t)z(x, t)) (z(x, t) - g(x, t)) &= 0.
\end{align*}
\] (20)

holds a.e. in \( \mathbb{R} \times [0, T] \). Here we assume the existence of a function \( g \), such that the final condition can be written as

\[ z(x, T) = g(x, T) \text{ for } x \in \mathbb{R}, \]

and the boundary conditions are

\[
\begin{align*}
z(x, t) &= g(x, t), & x &\to \infty, \\
z(x, t) &= g(x, t), & x &\to -\infty.
\end{align*}
\]

Next, we construct the discrete version of (20) in a large enough bounded domain \( [x_{\min}, x_{\max}] \times [0, T] \). For this purpose, let us define the finite difference operator:

\[
\mathcal{L}_h Z^m_n := \frac{a^m}{h^2} (Z^m_{n+1} - 2Z^m_n + Z^m_{n-1}) + \frac{b^m}{2h} (Z^m_{n+1} - Z^m_{n-1}) + c^m Z^m_n,
\]

where \( l = T/M \) and \( h = (x_{\max} - x_{\min})/N \) represent the discretization steps sizes in time and space, respectively. Moreover, we introduce the notation \( t^m = ml, m = 0, \ldots, M \) and \( x_n = x_{\min} + nh, n = 0, \ldots, N \) so that the finite differences mesh is \( \{(x_n, t^m), m = 0, \ldots, M, n = 0, \ldots, N\} \). Thus, the main goal is to construct an approximation \( \hat{Z}^m_n \approx z(x_n, t^m) \) at the mesh points by using a finite difference \( \theta \)-operator related to \( G_{x, t} \) which is given by

\[
\hat{G}^\theta_{h, l} \hat{Z}^m_n := \frac{\hat{Z}^{m+1}_n - \hat{Z}^m_n}{l} + (1 - \theta) \mathcal{L}_h \hat{Z}^{m+1}_n + \theta \mathcal{L}_h \hat{Z}^m_n. \] (21)

Thus, by applying \( \mathcal{L}_h \) to (21), we get

\[
\begin{align*}
l \hat{G}^\theta_{h, l} \hat{Z}^m_n &= \hat{Z}^{m+1}_n - \hat{Z}^m_n + \frac{(1 - \theta)l a^{m+1}}{h^2} (\hat{Z}^{m+1}_{n+1} - 2\hat{Z}^{m+1}_n + \hat{Z}^{m+1}_{n-1}) + \\
&+ \frac{\theta l a^{m+1}}{h^2} (\hat{Z}^m_{n+1} - 2\hat{Z}^m_n + \hat{Z}^m_{n-1}) + \frac{(1 - \theta)l b^{m+1}}{2h} (\hat{Z}^{m+1}_{n+1} - \hat{Z}^{m+1}_{n-1}) + \\
&+ \frac{\theta l b^{m+1}}{2h} (\hat{Z}^m_{n+1} - \hat{Z}^m_{n-1}) + (1 - \theta)l c^{m+1} \hat{Z}^{m+1}_n + \theta l c^m \hat{Z}^m_n.
\end{align*}
\] (22)
After some easy computations, we obtain a general form of the \(\theta\)-scheme discretization

\[
G^{\theta}_{h,l} Z_n = \begin{cases}
\left( -\frac{\theta}{h^2} \right) Z_{n-1} + \left( 1 + \frac{2\theta}{h^2} - \theta c_n \right) Z_n + \left( -\frac{\theta}{h^2} \right) Z_{n+1} \\
+ \left( (1-\theta) l_n^2 \right) Z_{n-1} + \left( 1 - (1-\theta) l_n^2 \right) Z_n + \left( (1-\theta) l_n^2 \right) Z_{n+1}.
\end{cases}
\]

(23)

Let us introduce the coefficients:

\[
A_n = \frac{la_n^m}{h^2} - \frac{lb_n^m}{2h}, \quad B_n = -\frac{2la_n^m}{h^2} + lc_n^m, \quad C_n = \frac{la_n^m}{h^2} + \frac{lb_n^m}{2h}.
\]

(24)

By replacing expressions (24) into (23) we can use the more compact notation for \(m = M-1, M-2, \ldots, 0\):

\[
l \hat{G}^{\theta}_{h,l} Z_n = \theta A_n^m Z_{n-1} - (1-\theta B_n^m) Z_n + \theta C_n^m Z_{n+1} +
\]

\[
+ (1-\theta) A_n^{m+1} Z_{n-1} + (1 + (1-\theta) B_n^{m+1}) Z_n + (1-\theta) C_n^{m+1} Z_{n+1}.
\]

(25)

Then, we can state a discrete version of problem (20) as follows: Find a finite sequence

\[
\{ Z_n^m : m = 0, 1, \ldots, M; n = 0, 1, \ldots, N \}
\]

such that

\[
\begin{align*}
\hat{G}^{\theta}_{h,l} Z_n^m &\leq 0, \\
Z_n^m - g(x_n, t^m) &\geq 0, \\
\left( \hat{G}^{\theta}_{h,l} Z_n^m \right)^t (Z_n^m - g(x_n, t^m)) &\geq 0.
\end{align*}
\]

(26)

holds for \(m = 0, 1, \ldots, M\) and \(n = 0, 1, \ldots, N\). Recall that we know the values

\[
\hat{Z}_0^m = g(x_0, t^m) \quad \text{and} \quad \hat{Z}_N^m = g(x_N, t^m)
\]

for \(m = 0, 1, \ldots, M\) and

\[
\hat{Z}_n^M = g(x_n, Ml)
\]

for \(n = 0, 1, \ldots, N\). Let \(Z^m\) be the \((N-1)\)-dimensional unknown vector at time \(t^m\), given by:

\[
Z^m = \begin{pmatrix} \hat{Z}_1^m, \hat{Z}_2^m, \ldots, \hat{Z}_{N-1}^m \end{pmatrix}.
\]
and the \((N - 1) \times (N - 1)\) tridiagonal matrix
\[
G_m = \begin{bmatrix}
B_1^m & C_1^m & 0 & \ldots & 0 & 0 \\
A_2^m & B_2^m & C_2^m & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & B_{N-2}^m & C_{N-2}^m \\
0 & 0 & 0 & \ldots & A_{N-1}^m & B_{N-1}^m
\end{bmatrix}
\]
Then we can write (26) in matrix form as:
\[
\begin{cases}
(id - \theta G_m)Z_m \leq b^{m+1}, \\
Z_m \geq g_m, \\
((id - \theta G_m)Z_m - b^{m+1})^t(Z_m - g_m) = 0
\end{cases}
\text{for } m = M - 1, M - 2, \ldots, 1, 0 \quad (27)
\]
where
\[
b^{m+1} = (id + (1 - \theta)G_{m+1})Z_{m+1} + (1 - \theta)A_1^{m+1}Z_0 + \theta A_1^mZ_0 + \ldots + \theta C_{N-1}^mZ_{N-1}^m + \theta C_m^mZ_m^m.
\]

3.1. Pricing American-style Options: A numerical approach to the Linear Complementarity Problem

Although pricing a European vanilla bond option with finite differences is certainly instructive in order to give an insight of which numerical method may be more efficient, it results not very practical in the real market situations because we are equipped with well-known closed-form solutions for the HW model. Therefore, we apply these schemes to American options, for which exact formulas are not available. In this early exercise setting, in order to avoid arbitrage, the option pricing models establish that at any time and for any value of the underlying, the option price cannot be less than its intrinsic value (which is the instantaneous received payoff if the option is exercised).

From a practical point of view, taking this condition into account is not very difficult when we take \(\theta = 0\) in (26), that is, in an explicit scheme. More precisely, after computing the predicted value of \(Z^m = b^{m+1}\) as in the European style option, the possibility of early exercise by comparing with the intrinsic value is analyzed and set
\[
\hat{Z}_n^m = \max \left( \hat{Z}_n^m, g(x_n, t^m) \right), \quad n = 1, \ldots, N - 1 \quad (28)
\]
Therefore, if we want to price American-style options, we need to construct previously the full grid containing the early exercise prices, \(g(x_n, t^m)\), and next the separate grid with European option prices, \(\hat{Z}_n^m\).
3.2. A Direct LU solver

Due to accuracy and computing time issues, we might prefer adopting an implicit scheme, that is, \(0 < \theta \leq 1\). In this case as in any implicit scheme, for each time layer \(m\) we have the full-problem (27) and the relationship above (28) requires the explicit knowledge of each \(\tilde{Z}_n^m\) already, which is not the case in an implicit scheme. In order to overcome this difficulty we use a LU decomposition and a modified backward substitution with a projection to solve the LCP (27) as discussed in [14, Sect. 4.1] and in [19, Chapter 4, pp. 189–190] where it is shown to be equivalent to the refined approach due to Brennan and Schwartz [3] for valuing American-style options.

In order to briefly recall it, we consider the notation in (27):

\[
A = \text{id} - \theta G, \quad x = Z^m, \quad b = b^{m+1}, \quad g = g^m,\tag{29}
\]

so that (27) becomes

\[
Ax \leq b, \quad x \geq g, \quad (Ax - b)^t(x - g) = 0.\tag{30}
\]

The latter LCP can be restated as solving the linear system \(Ax = b\) componentwise in such a way that the free boundary condition \(x \geq g\) is always satisfied.

Notice that, in the standard form, the solution of the system of linear equations using LU decomposition consists of the forward and the backward substitutions:

\[
Ly = b \quad \text{and} \quad Ux = y,
\]

where \(L\) is a lower triangular matrix which has ones in its diagonal and \(U\) is an upper triangular matrix.

The following algorithm allows to impose the aforementioned free side condition with a LU direct method by using the max-function in the backward substition \(Ux = y\) because it is easy to arrange for standard plain vanilla options with early exercise. First solve \(Ly\) for \(y\), as the traditional implementation formally sets for the forward substitution:

\[
\begin{align*}
y_1 &= b_1, \\
y_n &= b_n - \sum_{p=1}^{n-1} L_{n,p}y_p & \text{for} \ n = 2, \ldots, N - 1.
\end{align*}
\]

Then solve \(Ux = y\) for \(x\) preserving the condition \(x \geq g\) of the LCP (30). This can be done
by the modified back substitution:

\[
\begin{align*}
  x_{N-1} &= \max \left\{ h_{N-1}, \frac{y_{N-1}}{U_{N-1,N-1}} \right\}, \\
  x_n &= \max \left\{ h_n, \frac{1}{U_{n,n}} \left( y_n - \sum_{p=n+1}^{N-1} U_{n,p} x_p \right) \right\} \quad \text{for } n = N - 2, N - 3, \ldots, 1.
\end{align*}
\]

(31)

We remind that, by nature, this method is as efficient as solving a standard tridiagonal system of linear equations. Therefore, it is equally fast as the implementation for an implicit method made through this work for the less attractive case of European vanilla options on discount bonds.

3.3. Implementing the numerical method for Pricing American Bond Options

Now, consider the linear complementarity problem (16) and recall that we can write a solution \( w(x,t) \) of (16) by

\[
\begin{align*}
  w(x,t) &= e^{-\int_t^T \Omega(q) dq} u(x,t) \\
  A_{x,t} w(x,t) &= e^{-\int_t^T \Omega(q) dq} B_{x,t} u(x,t),
\end{align*}
\]

where \( B_{x,t} \) is a partial differential linear operator with time-independent coefficients. Since we can also write (16) as

\[
\begin{align*}
  \begin{cases}
    B_{x,t} u(x,t) \leq 0, \\
    u(x,t) - e^{\int_t^T \Omega(q) dq} g(x + \Omega(t), t) \geq 0, \\
    (B_{x,t} u(x,t)) \left( u(x,t) - e^{\int_t^T \Omega(q) dq} g(x + \Omega(t), t) \right) = 0,
  \end{cases}
\end{align*}
\]

(32)

we proceed as follows. First we construct the discrete \( \tilde{B}_{h,t}^\theta \) operator and as above we obtain the \( \theta \)-discrete scheme (27) where the matrix

\[
G^m = G = \begin{bmatrix}
B_1 & C_1 & 0 & \cdots & 0 & 0 \\
A_2 & B_2 & C_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & B_{N-2} & C_{N-2} \\
0 & 0 & 0 & \cdots & A_{N-1} & B_{N-1}
\end{bmatrix}
\]

for all \( m = 0,1,\ldots,M - 1 \). It allows us to solve numerically (27) obtaining \( \tilde{U}_n^m \) as an approximation of \( u(x_n,t^m) \) and next we take \( \tilde{W}_n^m = e^{-\int_{t^m}^{T_m} \Omega(q) dq} \tilde{U}_n^m \) as an approximation of \( w(x_n,t^m) \).
For vanilla put options written on pure discount bonds we assume the following boundary conditions for the derivative price function \( w(x,t) \),

\[
\begin{align*}
w(x,t) &= g(x + \Omega(t), t), \quad x \to \infty, \quad (33) \\
w(x,t) &= g(x + \Omega(t), t), \quad x \to -\infty. \quad (34)
\end{align*}
\]

Although initially the mathematical problem is posed in an unbounded domain, in order to apply numerical methods we have to truncate the domain to a bounded one (localization procedure), thus, the boundary conditions become

\[
\begin{align*}
w(x_{\text{max}},t) &= h(x_{\text{max}} + \Omega(t), t) \sim \lim_{x \to \infty} h(x,t) = K, \quad (35) \\
w(x_{\text{min}},t) &= h(x_{\text{min}} + \Omega(t), t) \sim \lim_{x \to -\infty} h(x,t) = 0, \quad (36)
\end{align*}
\]

with \( t \in [0, T] \).

**Remark 1.** In practice we take a symmetric computational domain with \( x_{\text{min}} = -x_{\text{max}} \) and \( x_{\text{max}} \) large enough. As suggested by Cairns [5] and Daglish [6] we fix \( x_{\text{max}} = \frac{5\sigma}{\sqrt{2a}} \), see Appendix A for a justification.

4. Some numerical experiments

The explicit finite difference (EFD henceforth) and the implicit Crank-Nicolson (CN for short) are different from the trinomial tree approach of Hull and White [11] (HWT hereafter), as we have analyzed in the previous sections.

Due to the fact that with the HWT algorithm a previous step of forward induction is mandatory, this method results to be slower than the EFD algorithm even if we use the Arrow-Debreu prices to avoid backward recursion as suggested in [16, Chap. 2, pp. 23] for vanilla European-style bond options.

**European-style Bond Options**

In this subsection we present some numerical results to demonstrate the performance of the Crank-Nicolson scheme in the context of the options with no early exercise where analytical solutions exist. In particular, we compare the efficiency, accuracy and speed of convergence of the CN, EFD and HWT methods.

As first examples, we consider the pricing of two-year and three-year vanilla put options, written on a five-year discount bond. We assume the values \( a = 3\% \) and \( \sigma = 0.75\% \), which
Figure 1: Initial zero-rate curve on (notional) discount bonds without a default risk on June 29, 2012 (end-of-month data) as kindly provided by Deutsche Bundesbank. All rates are expressed with continuous compounding.

<table>
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<tr>
<th>Maturity, x</th>
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<td>0.03%</td>
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<table>
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</thead>
<tbody>
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<td>Zero Rate, $R^o(x)$</td>
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<td>0.91%</td>
<td>1.13%</td>
<td>1.35%</td>
<td>1.54%</td>
<td>1.73%</td>
</tr>
</tbody>
</table>

are very close to the ones recently observed on the markets. In our analysis we also use the discount curve given in the table on Figure 1 and the lowest dimensional consistent family with the model (18) to interpolate these data.

For EFD and CN methods, both option and closed-form formulas for the discount $S$-bond prices

$$P(x_n, t^m, S) := P^m_m(S),$$

are computed on the same grid.

The payoff condition for the put bond option at maturity date $t^M$ implies the following final condition at grid points:

$$W^M_n = \left( K - P^M_n(S) \right)^+, \quad n = 1, \ldots, N - 1. \quad (37)$$

On the other hand, for the trinomial tree approach HWT, we price European-style options by using the elementary Arrow-Debreu prices $Q^M_n$ following [16, Sect. 2.2.4, pp. 22–23] and the well-known fundamental formula

$$\sum_{n=0}^N Q^M_n W^M_n.$$ 

Firstly, as illustrated by Figures 2 and 3 for European-style options pricing, CN method outperforms the other ones, EFD and HWT, if smart ratios between the time and space steps $h$ and $l$ are chosen. More precisely, Figure 2 shows how the relative pricing errors for the CN method varies with lattice size for different choices between $h$ and $l$. It can be observed that the CN method implemented with linear relationship $h_k = k\sigma l$, henceforth CN(L), results better in terms of accuracy than if $h_{SR} = \sigma \sqrt{3l}$ is taken, referred as CN(SR) hereafter. Note that EFD and HWT methods are conditionally stable whenever we take
Thus, we have also included the CN(SR) implementation of the Crank-Nicolson method for an easy comparison with the EFD and HWT methods.

Secondly, the results for European-style options in Figure 4 and Figure 6 show that both EFD method and CN(SR) implementation are marginally better than HWT approach in terms of accuracy. We also note that almost the same speed of convergence is observed with CN(SR) and EFD methods provided that we choose the convenient ratio $\sigma^2 \varphi_1 = 1/3$ to ensure the stability of the EFD scheme, as shown in Figure 5. Again, the CN(L) implementation, here we take $h \sqrt{2} = \sqrt{2} \sigma l$ as in [6], outperforms their counterparts EFD, HWT and CN(SR) implementation in terms of accuracy and speed of convergence.

Thirdly, concerning to time consumption performance, we report in the table on Figure 7 the computing times for the four implementations. EFD algorithm is by far faster than the HWT algorithm while both Crank-Nicolson implementations are slower than the EFD scheme as expected due to linear system solving computational cost, although it has been optimized by using vector storage and computing for matrices.

Last but not least, these first experiments suggest that the CN(L) implementation is clearly the best choice. However, we have to be cautious as the CN method requires a costly (in time) solution of the tridiagonal system of equations which we have implemented by means of the Thomas algorithm. However, as seen in Figure 3, the CN(L) implementation is capable to perform an $O(10^{-7})$ accuracy, with a MATLAB consumption time of less than a second, much better than the rest of implementations taking a similar time consumption into consideration.

Next, concerning to American-style bond options, we report in the table on Figure 4 the results for all four implementations. The results for the CN(L) method are the best in terms of the speed of convergence and compatible with the results reported by several commercial black boxes. As can be seen, the results for the rest of the implementations, namely EFD, CN(SR) and HWT, are slightly similar with less time consumption for both EFD and CN(SR) finite difference methods (see American-style options block on Figure 7). We point out that the difference between orders of convergence and orders of computational time required becomes very apparent in the table for the American options on Figure 7. All implementations reduce errors as $\Delta t = l$ declines. However, both the HWT method and the finite difference methods in its approaches EFD and CN(SR) require rapidly increasing amounts of time to achieve the same accuracy that the CN(L) approach. For example, in the two year case to achieve the limit value $84.81 \text{ CL(N)}$ requires $l = 0.100$ and 0.11
seconds, while HWT, EFD or CN(SR) require \( t = 0.0005 \) and 9.65, 1.06 and 0.61 seconds, respectively.

Next, we compare results when the direct LU and the projected SOR algorithms are applied for the discretized LCP problem for both CN implementations CN(SR) and CN(L). The stopping criterion for the iterative pSOR was chosen to be \( O(10^{-14}) \). The overrelaxation parameter \( \omega \) in the pSOR method was optimized using the approximations (39) and (40) in [13] so that iterations would converge as fast as possible. The method pSOR is well-known and widely applied for pricing American-style equity options; see, for example, [19, pp. 155–158].

Table in Figure 8 shows the speed of convergence of American put prices to the truth solution of the problem for just the CN implementations. We remark that the same parameters \((a, \sigma)\) and initial curve data (1) are used. The block pSOR in Figure 8, exhibits the maximum number of iterations and the median of the overrelaxation parameter, columns ‘maxiter’ and ‘\( w \)’ respectively, to achieve the stopping criterion aforementioned above. These statistics try to summarize the whole distribution of these control parameters across the sequence of LCPs solved at time layers \( m = M - 1, \ldots, 1 \). We also report the price approximant.

First, with regard to the pSOR algorithm, it is notorious that for the CN(L) implementation much more iterations and recalibration of the \( \omega \) control parameter are needed for finer grids. Second, we point out the undesired oscillations for the price approximant produced when the CN(L) implementation with smaller time-steps is considered. Last but not least, we remark the goodness of the direct LU algorithm which converges very quickly for the CN(L), and the absence of unsteady behaviours for the CN(SR) implementation as well.

Furthermore, in the numerical experiments we compared the required CPU times between the direct LU and the iterative pSOR methods. The Table in the Figure 9 indicates that the direct LU algorithm is several times faster on coarser grids, so that CN(SR) implementation, and tens of times faster on finer grids, which corresponds to CN(L) implementation. Additionally, it is difficult to find an optimal stopping criterion and overrelaxation parameter for the projected SOR method while the direct LU algorithm is parameter free and, thus, much easier to use.

Finally, since analytical solutions are unknown, we define a convergence indicator \( \epsilon \) to compute the following ratios of the numerical approximants of the consecutive meshes

\[
\epsilon_k = \frac{|p(l_k) - p(l_{k-1})|}{p(l_k)} , \quad l = [0.5 \ 0.2 \ \ldots \ 0.001 \ 0.0005].
\]
The plots in Figure 10 clearly indicate that the CN(L) implementation in combination with the direct LU solver outperforms another approaches.

5. Conclusions

In this work we present an implicit Crank-Nicolson scheme for complementarity problem arising from pricing American put options on a pure discount bond under the HW model. One of our implementations, namely CN(L), in contrast to more traditional explicit or tree-based implementations, has the advantage of being second-order accurate in time steps being unconditionally stable, as it is well known for CN methods and extremely exploited in the equity options pricing literature.

We have presented a methodology which is suitable to adapt for higher-order methods and other multifactor models. Moreover, it results in a more efficient pricing of American bond options when a direct LU algorithm is used to solve the related linear complementarity problem.

On the other hand, for practitioners computational speed is a key feature. We note the very promising numerical costs provided by the Crank-Nicolson implementation CN(L) in combination with the direct LU algorithm to solve the free-boundary problem when the contract under consideration has early exercise capabilities.

Acknowledgements

This work has been supported by the grant PROMETEOII/2013/015 from the Education Council of Generalitat Valenciana.
Figure 2: CN relative pricing errors versus lattice size for different relationships between $h$ and $l$. By $h_{SR}$ we mean a relationship of the square-root type $h = \sigma \sqrt{3l}$. The rest of relationships are of the type $h_k = k\sigma l$.

(Vanilla 3yr put on 5yr discount bond: $a = 3\%$, $\sigma = 0.75\%$, $K = 0.975$.)

Figure 3: Relative pricing errors versus calculation times. CN(L) means Crank-Nicolson method with the linear relationship between steps $h_{\sqrt{2}} = \sqrt{2}\sigma l$ whereas CN(SR) denotes a square-root relationship $h_{SR}$

(European vanilla 3yr put on 5yr discount bond: $a = 3\%$, $\sigma = 0.75\%$, $K = 0.975$.)
Figure 4: Prices (in basis points of the notional) for put options on 5yr discount bond.

<table>
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<th>$l$</th>
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<th></th>
<th>American-style</th>
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<td>CN(L)</td>
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<td>47.78</td>
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Note: We recall that HWT refers to the Hull-White trinomial tree lattice method, and EFD and CN(SR) to the conditionally stable explicit method and the Crank-Nicolson method with square-root relationship between steps, respectively. Therefore, for all these three methods, $h = h_{SR}$ it is assumed as in [11]. CN(L) refers to the Crank-Nicolson method with the linear relation $h \sqrt{2} = \sqrt{2} \sigma \tau$ between steps taken into account. For the EFD, CN(SR) and CN(L) methods we set boundaries at $\pm 5 \frac{\sigma}{\sqrt{2} \tau}$ following [5] and [6].
Figure 5: Stability Issues for the EFD Method with different values of the stability parameter $\kappa = \sigma^2 \varrho_1$ (European vanilla 2yr put on 5yr discount bond: $a = 3\%$, $\sigma = 0.75\%$, $K = 0.95$.)

Figure 6: Relative pricing errors of European vanilla 3yr put on 5yr discount bond.
Figure 7: Calculation times in seconds, running MATLAB 7.14.0.739 on an Intel Core i5-3337U @ 1.80GHz computer. (Vanilla 2yr put on 5yr discount bond: $a = 3\%$, $\sigma = 0.75\%$, $K = 0.95$.)

<table>
<thead>
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Figure 8: Prices (in basis points of the notional) for American put options on 5yr discount bond.

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<th>( \omega )</th>
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Figure 9: Calculation times in seconds, running MATLAB 7.14.0.739 on an Intel Core i5-3337U @ 1.80GHz computer. (American 3yr put on 5yr discount bond: \( a = 3\%\), \( \sigma = 0.75\%\), \( K = 0.95\).)

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Figure 10: Convergence of American 3yr put on 5yr discount bond.
Appendix A. A Brief Note on Boundary Conditions

First, we are concerned with how accurate the various methods are, both relative to each other and to the roughness lattice. The state variable

\[ x = r - \Omega(t), \]

dependently ranges from \(-\infty\) to \(+\infty\) and we restrict this to the range \([-x_{\text{max}}, x_{\text{max}}]\) with \(x_{\text{max}} = 5\frac{\sigma}{\sqrt{2a}}\) in both the EFD and CN method.

Second, we recall the reduced PDE for the \(w(x,t)\) function

\[ w_t + \frac{1}{2}\sigma^2 w_{xx} - axw_x + (x + \Omega(t)) w = 0. \]

This PDE may be viewed as the deterministic counterpart of a stochastic model where, under the equivalent martingale measure \(Q\), the spot interest rate \(r\) is generated by

\[
\begin{align*}
\text{d}X_t &= -aX_t \text{d}t + \sigma \text{d}W \\
\text{r}_t &= X_t + \Omega(t)
\end{align*}
\]

This stochastic process for the underlying variable \(X\) is determined only by the parameters \(a\) and \(\sigma\) and is independent of the function \(\Omega(t)\). The process that \(X_t\) is assumed to follow is the well-known Ornstein-Uhlenbeck process, with conditional moments at \(T > t\):

\[
\begin{align*}
\mathbb{E}[X_T|X_t = x] &= xe^{-a(T-t)} \\
\text{Var}[X_T|X_t = x] &= \frac{\sigma^2}{2a} \left(1 - e^{-2a(T-t)}\right)
\end{align*}
\]

which means that for \(a > 0\) and for large \(T\) the mean of \(X\) falls to 0 and the variance tends to \(\frac{\sigma^2}{2a}\).

Therefore, although that election made for the pair of limits \([-x_{\text{max}}, x_{\text{max}}]\) may be certainly subjective, we remark that in both CN and EFD implementations correspond to \(\pm 5\) standard deviations of the stationary distribution of \(X_t\) (that is, \(\pm 5\frac{\sigma}{\sqrt{2a}}\)).